

A HISTORY
OF
GREEK MATHEMATICS

BY

SIR THOMAS HEATH

K.C.B., K.C.V.O., F.R.S.

SC.D. CAMB. ; HON. D.SC. OXFORD

HONORARY FELLOW (FORMERLY FELLOW) OF TRINITY COLLEGE, CAMBRIDGE

‘. . . An independent world,
Created out of pure intelligence.’
WORDSWORTH.

VOLUME II
FROM ARISTARCHUS TO DIOPHANTUS

CONTENTS OF VOL II

XII. ARISTARCHUS OF SAMOS	PAGES 1-15
XIII. ARCHIMEDES	16-109
Traditions	
(a) Astronomy	17-18
(s) Mechanics	18
Summary of main achievements	19-20
Character of treatises	20-22
List of works still extant	22-23
Traces of lost works	23-25
The text of Archimedes	25-27
Contents of <i>The Method</i>	27-34
<i>On the Sphere and Cylinder</i> , I, II	34-50
Cubic equation arising out of II. 4	43-46
(i) Archimedes's own solution	45-46
(ii) Dionysodorus's solution	46
(iii) Diocles's solution of original problem	47-49
<i>Measurement of a Circle</i>	50-56
<i>On Conoids and Spheroids</i>	56-64
<i>On Spirals</i>	64-75
<i>On Plane Equilibriums</i> , I, II	75-81
<i>The Sand-reckoner</i> (<i>Psammites</i> or <i>Arenarius</i>)	81-85
<i>The Quadrature of the Parabola</i>	85-91
<i>On Floating Bodies</i> , I, II	91-97
The problem of the crown	92-94
Other works	
(a) The Cattle-Problem	97-98
(s) On semi-regular polyhedra	98-101
(γ) The <i>Liber Assumptorum</i>	101-103
(δ) Formula for area of triangle	103
Eratosthenes	104-109
Measurement of the Earth	106-108
XIV. CONIC SECTIONS. APOLLONIUS OF PERGA	110-196
A. HISTORY OF CONICS UP TO APOLLONIUS	
Discovery of the conic sections by Menaechmus	110-111
Menaechmus's probable procedure	111-116

Oxford University Press, Amen House, London E.C.4

GLASGOW NEW YORK TORONTO MELBOURNE WELLINGTON

BOMBAY CALCUTTA MADRAS KARACHI LAHORE DACCA

CAPE TOWN SALISBURY NAIROBI IBADAN ACCRA

KUALA LUMPUR HONG KONG

510.9

N4380

V.2

Cop.3

CONTENTS OF VOL II

XII. ARISTARCHUS OF SAMOS	PAGES 1-15
XIII. ARCHIMEDES	16-109
Traditions	
(a) Astronomy	17-18
(β) Mechanics	18
Summary of main achievements	
Character of treatises	19-20
List of works still extant	20-22
Traces of lost works	22-23
The text of Archimedes	23-25
Contents of <i>The Method</i>	25-27
<i>On the Sphere and Cylinder</i> , I, II	27-34
Cubic equation arising out of II. 4	34-50
(i) Archimedes's own solution	43-46
(ii) Dionysodorus's solution	45-46
(iii) Diocles's solution of original problem	46
<i>Measurement of a Circle</i>	
<i>On Conoids and Spheroids</i>	47-49
<i>On Spirals</i>	50-56
<i>On Plane Equilibriums</i> , I, II	56-64
<i>The Sand-reckoner</i> (<i>Psammites</i> or <i>Arenarius</i>)	64-75
<i>The Quadrature of the Parabola</i>	75-81
<i>On Floating Bodies</i> , I, II	81-85
The problem of the crown	85-91
Other works	
(a) The Cattle-Problem	91-97
(β) On semi-regular polyhedra	97-98
(γ) <i>The Liber Assumptorum</i>	98-101
(δ) Formula for area of triangle	101-103
Eratosthenes	
Measurement of the Earth	103
	104-109
	106-108
XIV. CONIC SECTIONS. APOLLONIUS OF PERGA	110-196
A. HISTORY OF CONICS UP TO APOLLONIUS	
Discovery of the conic sections by Menaechmus	110-111
Menaechmus's probable procedure	111-116

CONTENTS

XIV. CONTINUED.

B. APOLLONIUS OF PERGA		PAGES	126-196
The text of the <i>Conics</i>			126-128
Apollonius's own account of the <i>Conics</i>			128-133
Extent of claim to originality			132-133
Great generality of treatment			133
Analysis of the <i>Conics</i>			133-175
Book I			133-148
Conics obtained in the most general way from oblique cone			134-138
New names, 'parabola', 'ellipse', 'hyperbola'			138-139
Fundamental properties equivalent to Cartesian equations			139-141
Transition to new diameter and tangent at its extremity			141-147
First appearance of principal axes			147-148
Book II			148-150
Book III			150-157
Book IV			157-158
Book V			158-167
Normals as maxima and minima			159-163
Number of normals from a point			163-164
Propositions leading immediately to determination of <i>evolutes</i> of conic			164-166
Construction of normals			166-167
Book VI			167-168
Book VII			168-174
Other works by Apollonius			175-194
(a) <i>On the Cutting-off of a Ratio</i> (λόγου ἀποτομή), two Books			175-179
(β) <i>On the Cutting-off of an Area</i> (χωρίου ἀποτομή), two Books			179-180
(γ) <i>On Determinate Section</i> (διωρισμένη τομή), two Books			180-181
(δ) <i>On Contacts or Tangencies</i> (ἐπαφαί), two Books			181-185
(ε) <i>Plane Loci</i> , two Books			185-189
(ζ) <i>Nεύσεις</i> (Vergings or Inclinations), two Books			189-192
(η) <i>Comparison of dodecahedron with icosahedron</i>			192
(θ) <i>General Treatise</i>			192-193
(ι) <i>On the Cochlias</i>			193
(κ) <i>On Unordered Irrationals</i>			193
(λ) <i>On the Burning-mirror</i>			194
(μ) <i>Ὠκυρόκιον</i>			194
Astronomy			195-196

XV. THE SUCCESSORS OF THE GREAT GEOMETERS . 197-234

Nicomedes
Diocles

Geminus	PAGES 222-234
Attempt to prove the Parallel-Postulate	227-230
On <i>Meteorologica</i> of Posidonius	231-232
<i>Introduction to the Phaenomena</i> attributed to Geminus	232-234
 XVI. SOME HANDBOOKS	235-244
Cleomedes, <i>De motu circulari</i>	235-238
Nicomachus	238
Theon of Smyrna, <i>Expositio rerum mathematicarum ad legendum Platonem utilium</i>	238-244
 XVII. TRIGONOMETRY: HIPPARCHUS, MENELAUS, PTO- LEMY	245-297
Theodosius	245-246
Works by Theodosius	246
Contents of the <i>Sphaerica</i>	246-252
No actual trigonometry in Theodosius	250-252
The beginnings of trigonometry	252-253
Hipparchus	253-260
The work of Hipparchus	254-256
First systematic use of trigonometry	257-259
Table of chords	259-260
Menelaus	260-273
The <i>Sphaerica</i> of Menelaus	261-273
(α) 'Menelaus's theorem' for the sphere	266-268
(β) Deductions from Menelaus's theorem	268-269
(γ) Anharmonic property of four great circles through one point	269-270
(δ) Propositions analogous to Eucl. VI. 3	270
Claudius Ptolemy	273-297
The <i>Μαθηματικὴ σύνταξις</i> (Arab. <i>Almagest</i>)	273-286
Commentaries	274
Translations and editions	274-275
Summary of contents	275-276
Trigonometry in Ptolemy	276-286
(α) Lemma for finding $\sin 18^\circ$ and $\sin 36^\circ$	277-278
(β) Equivalent of $\sin^2 \theta + \cos^2 \theta = 1$	278
(γ) 'Ptolemy's theorem', giving the equivalent of $\sin(\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi$	278-280
(δ) Equivalent of $\sin^2 \frac{1}{2} \theta = \frac{1}{2} (1 - \cos \theta)$	280-281
(ϵ) Equivalent of $\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$	281
(ζ) Method of interpolation based on formula $\sin a / \sin \beta < a / \beta (\frac{1}{2} \pi > a > \beta)$	281-282
(η) Table of chords	283
(θ) Further use of proportional increase	283-284
(i) Plane trigonometry in effect used	284
Spherical trigonometry: formulæ in solution of spherical triangles	284-285

XVIII. MENSURATION: HERON OF ALEXANDRIA. PAGES 298	
Controversies as to Heron's date	298
Character of works	307
List of treatises	308
Geometry	310
(a) Commentary on Euclid's <i>Elements</i>	310
(β) The <i>Definitions</i>	314
Mensuration	316
The <i>Metrica</i> , <i>Geometrica</i> , <i>Stereometrica</i> , <i>Geodæsia</i> , <i>Mensurae</i>	316-3
Contents of the <i>Metrica</i>	320-3
Book I. Measurement of areas	320-3
(a) Area of scalene triangle	320-3
Proof of formula $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$	321-3
(β) Method of approximating to the square root of a non-square number	323-3
(γ) Quadrilaterals	326-32
(δ) Regular polygons with 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12 sides	326-32
(ε) The circle	326-32
(ζ) Segment of a circle	326-32
(η) Ellipse, parabolic segment, surface of cylinder, right cone, sphere and segment of sphere	330-33
Book II. Measurement of volumes	331-33
(a) Cone, cylinder, parallelepiped (prism), pyramid and frustum	331-33
(β) Wedge-shaped solid (<i>βαμίσκος</i> or <i>σφηνίσκος</i>)	332-33
(γ) Frustum of cone, sphere, and segment thereof	334
(δ) Anchor-ring or tore	334-335
(ε) The two special solids of Archimedes's 'Method'	335
(ζ) The five regular solids	335
Book III. Divisions of figures	336-343
Approximation to the cube root of a non-cube number	341-342
Quadratic equations solved in Heron	344
Indeterminate problems in the <i>Geometrica</i>	344
The <i>Dioptra</i>	345-346
The <i>Mechanics</i>	346-352
Aristotle's Wheel	347-348
The parallelogram of velocities	348-349
Motion on an inclined plane	349-350
On the centre of gravity	350-351
The five mechanical powers	351
Mechanics in daily life: queries and answers	351-352
Problems on the centre of gravity, &c.	352
The <i>Catoptrica</i>	352-354
Heron's proof of equality of angles of incidence and reflection	352-354

The <i>Synagoge</i> or <i>Collection</i>	PAGES 357-439
(a) Character of the work; wide range	357-358
(β) List of authors mentioned	358-360
(γ) Translations and editions	360-361
(δ) Summary of contents	361-439
Book III. Section (1). On the problem of the two mean proportionals	361-362
Section (2). The theory of means	363-365
Section (3). The 'Paradoxes' of Erycinus	365-368
Section (4). The inscribing of the five regular solids in a sphere	368-369
Book IV. Section (1). Extension of theorem of Pythagoras	369-371
Section (2). On circles inscribed in the <i>ἄρβηλος</i> ('shoemaker's knife')	371-377
Sections (3), (4). Methods of squaring the circle and trisecting any angle	377-386
(a) The Archimedean spiral	377-379
(β) The conchoid of Nicomedes	379
(γ) The <i>Quadratrix</i>	379-382
(δ) Digression: a spiral on a sphere	382-385
Trisection (or division in any ratio) of any angle	385-386
Section (5). Solution of the <i>νέυσις</i> of Archimedes, <i>On Spirals</i> , Prop. 8, by means of conics	386-388
Book V. Preface on the sagacity of Bees	389-390
Section (1). Isoperimetry after Zenodorus	390-393
Section (2). Comparison of volumes of solids having their surfaces equal. Case of sphere	393-394
Section (3). Digression on semi-regular solids of Archimedes	394
Section (4). Propositions on the lines of Archimedes, <i>On the Sphere and Cylinder</i>	394-395
Section (5). Of regular solids with surfaces equal, that is greater which has more faces	395-396
Book VI.	396-399
Problem arising out of Euclid's <i>Optics</i>	397-399
Book VII. On the 'Treasury of Analysis'	399-427
Definition of Analysis and Synthesis	400-401
List of works in the 'Treasury of Analysis'	401
Description of the treatises	401-404
Anticipation of Guldin's Theorem	403
Lemmas to the different treatises	404-426
(a) Lemmas to the <i>Sectio rationis</i> and <i>Sectio spatii</i> of Apollonius	404-405
(β) Lemmas to the <i>Determinate Section</i> of Apollonius	405-412
(γ) Lemmas on the <i>Νέυσις</i> of Apollonius	412-416
(δ) Lemmas on the <i>On Contacts</i> of Apollonius	416-417
(e) Lemmas to the <i>Plane Loci</i> of Apollonius	417-419
(f) Lemmas to the <i>Conics</i> of Apollonius	419-426

XIX. CONTINUED.

Book VIII (<i>continued</i>)	
The inclined plane	PAGES
Construction of a conic through five points	
Given two conjugate diameters of an ellipse, to find the axes	
Problem of seven hexagons in a circle	
Construction of toothed wheels and indented screws	

XX. ALGEBRA: DIOPHANTUS OF ALEXANDRIA

Beginnings learnt from Egypt	4
'Hau'-calculations	
Arithmetical epigrams in the Greek Anthology	4
Indeterminate equations of first degree	
Indeterminate equations of second degree before Diophantus	44
Indeterminate equations in Heronian collections	44
Numerical solution of quadratic equations	
Works of Diophantus	44
The <i>Arithmetica</i>	44
The seven lost Books and their place	44
Relation of 'Porisms' to <i>Arithmetica</i>	45
Commentators from Hypatia downwards	
Translations and editions	45
Notation and definitions	45
Sign for unknown ($= x$) and its origin	45
Signs for powers of unknown &c.	45
The sign (Λ) for <i>minus</i> and its meaning	45
The methods of Diophantus	45
I. Diophantus's treatment of equations	462
(A) Determinate equations	462
(1) Pure determinate equations	462
(2) Mixed quadratic equations	463
(3) Simultaneous equations involving quadratics	
(4) Cubic equation	
(B) Indeterminate equations	
(a) Indeterminate equations of the second degree	466
(1) Single equation	466
(2) Double equation	468
1. Double equations of first degree	469
2. Double equations of second degree	472
(b) Indeterminate equations of degree higher than second	
(1) Single equations	473
(2) Double equations	473
II. Method of limits	475
III. Method of approximation to limits	476
Porisms and propositions in the <i>Arithmetica</i>	477

XXI. COMMENTATORS AND BYZANTINES	PAGES 518-555
Serenus	519-526
(a) <i>On the Section of a Cylinder</i>	519-522
(β) <i>On the Section of a Cone</i>	522-526
Theon of Alexandria	526-528
Commentary on the <i>Syntaxis</i>	526-527
Edition of Euclid's <i>Elements</i>	527-528
Edition of the <i>Optics</i> of Euclid	528
Hypatia	528-529
Porphyry. Iamblichus	529
Proclus	529-537
Commentary on Euclid, Book I	530-535
(a) Sources of the Commentary	530-532
(β) Character of the Commentary	532-535
<i>Hypotyposis of Astronomical Hypotheses</i>	535-536
Commentary on the <i>Republic</i>	536-537
Marinus of Neapolis	537-538
Domninus of Larissa	538
Simplicius	538-540
Extracts from Eudemus	539
Eutocius	540-541
Anthemius of Tralles	541-543
<i>On burning-mirrors</i>	541-543
The Papyrus of Akhmīm	543-545
<i>Geodaesia</i> of 'Heron the Younger'	545
Michael Psellus	545-546
Georgius Pachymeres	546
Maximus Planudes	546-549
Extraction of the square root	547-549
Two problems	549
Manuel Moschopoulos	549-550
Nicolas Rhabdas	550-554
Rule for approximating to square root of a non-square number	553-554
Ioannes Pediasimus	554
Barlaam	554-555
Isaac Argyrus	555

APPENDIX. On Archimedes's proof of the subtangent-property of a spiral	556-561
---	---------

INDEX OF GREEK WORDS	563-569
--------------------------------	---------

ENGLISH INDEX	570-586
-------------------------	---------

XII

ARISTARCHUS OF SAMOS

HISTORIANS of mathematics have, as a rule, given too little attention to Aristarchus of Samos. The reason is no doubt that he was an astronomer, and therefore it might be supposed that his work would have no sufficient interest for the mathematician. The Greeks knew better; they called him Aristarchus 'the mathematician', to distinguish him from the host of other Aristarchuses; he is also included by Vitruvius among the few great men who possessed an equally profound knowledge of all branches of science, geometry, astronomy, music, &c.

'Men of this type are rare, men such as were, in times past, Aristarchus of Samos, Philolaus and Archytas of Tarentum, Apollonius of Perga, Eratosthenes of Cyrene, Archimedes and Scopinas of Syracuse, who left to posterity many mechanical and gnomonic appliances which they invented and explained on mathematical (lit. 'numerical') principles.'¹

That Aristarchus was a very capable geometer is proved by his extant work *On the sizes and distances of the Sun and Moon* which will be noticed later in this chapter: in the mechanical line he is credited with the discovery of an improved sun-dial, the so-called *σκάφη*, which had, not a plane, but a concave hemispherical surface, with a pointer erected vertically in the middle throwing shadows and so enabling the direction and the height of the sun to be read off by means of lines marked on the surface of the hemisphere. He also wrote on vision, light and colours. His views on the latter subjects were no doubt largely influenced by his master, Strato of Lampsacus; thus Strato held that colours were emanations from bodies, material molecules, as it were, which imparted to

ARISTARCHUS OF SAMOS

stamping the air with impressions like themselves, as it were that 'colours in darkness have no colouring', and that 'light is the colour impinging on a substratum'.

Two facts enable us to fix Aristarchus's date approximately. In 281/280 B.C. he made an observation of the summer solstice; and a book of his, presently to be mentioned, was published before the date of Archimedes's *Psammites* or *Sand-reckoner*, a work written before 216 B.C. Aristarchus, therefore, probably lived *circa* 310-230 B.C., that is, he was older than Archimedes by about 25 years.

To Aristarchus belongs the high honour of having been the first to formulate the Copernican hypothesis, which was then abandoned again until it was revived by Copernicus himself. His claim to the title of 'the ancient Copernicus' is still, in my opinion, quite unshaken, notwithstanding the ingenious and elaborate arguments brought forward by Schiaparelli to prove that it was Heraclides of Pontus who first conceived the heliocentric idea. Heraclides is (along with one Ecphantus, a Pythagorean) credited with having been the first to hold that the earth revolves about its own axis every 24 hours, and he was the first to discover that Mercury and Venus revolve, like satellites, about the sun. But though this proves that Heraclides came near, if he did not actually reach, the hypothesis of Tycho Brahe, according to which the earth was in the centre and the rest of the system, the sun with the planets revolving round it, revolved round the earth, it does not suggest that he moved the earth away from the centre. The contrary is indeed stated by Aëtius, who says that 'Heraclides and Ecphantus make the earth move, *not in the sense of translation*, but by way of turning on an axle, like a wheel, from west to east, about its own centre'.¹ None of the champions of Heraclides have been able to meet this positive statement. But we have conclusive evidence in favour of the claim of Aristarchus; indeed, ancient testimony is unanimous on the point. Not only does Plutarch tell us that Cleanthes held that Aristarchus ought to be indicted for impiety, but

contemporary, Archimedes. In the *Sand-reckoner* Archimedes has this passage.

‘You [King Gelon] are aware that “universe” is the name given by most astronomers to the sphere the centre of which is the centre of the earth, while its radius is equal to the straight line between the centre of the sun and the centre of the earth. This is the common account, as you have heard from astronomers. But Aristarchus brought out *a book consisting of certain hypotheses*, wherein it appears, as a consequence of the assumptions made, that the universe is many times greater than the “universe” just mentioned. His hypotheses are that *the fixed stars and the sun remain unmoved, that the earth revolves about the sun in the circumference of a circle, the sun lying in the middle of the orbit*, and that the sphere of the fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface.’

(The last statement is a variation of a traditional phrase, for which there are many parallels (cf. Aristarchus’s Hypothesis 2 ‘that the earth is in the relation of a point and centre to the sphere in which the moon moves’), and is a method of saying that the ‘universe’ is infinitely great in relation not merely to the size of the sun but even to the orbit of the earth in its revolution about it; the assumption was necessary to Aristarchus in order that he might not have to take account of parallax.)

Plutarch, in the passage referred to above, also makes it clear that Aristarchus followed Heraclides in attributing to the earth the daily rotation about its axis. The bold hypothesis of Aristarchus found few adherents. Seleucus, of Seleucia on the Tigris, is the only convinced supporter of it of whom we hear, and it was speedily abandoned altogether, mainly owing to the great authority of Hipparchus. Nor do we find any trace of the heliocentric hypothesis in Aristarchus’s extant work *On the sizes and distances of the Sun and Moon*. This is presumably because that work was

covered that the apparent diameter of the sun is $1/720$ th part of the zodiac circle, that is to say, $1/720$ th of 360° . We do not know how he arrived at this pretty accurate value, but, as he is credited with the invention of the *gnomon*, he must have used this instrument for the purpose. But the discovery must apparently have been later than the time of his treatise *On sizes and distances*, for the value of the angle is there assumed to be 2° (Hypothesis 6). That Aristarchus came to assume a value so excessive is unusual, but the mathematics of his treatise is not dependent on the value taken, 2° may have been assumed merely for illustration; or it may have been a guess at the true value of the diameter made before he had thought of attempting to measure it. Aristarchus assumed that the angular diameters of the sun and moon at the centre of the earth are equal.

The method of the treatise depends on the justly famous 'hypothesis' which is Aristarchus's third 'hypothesis', that 'when the moon appears to us halved, the great circle which divides the earth and the bright portions of the moon is in the direct line of sight'; the effect of this (since the moon receives its light from the sun), is that at the time of the dichotomy the sun, moon and earth form a triangle right-angled at the centre of the moon. Two other assumptions were made: first, an estimate of the size of the angle of the latitudes at the centre of the earth at the moment of dichotomy; Aristarchus assumed (Hypothesis 4) to be 'less than a quadrant by one-thirtieth of a quadrant', i.e. 87° , against the more accurate estimate, the true value being $89^\circ 50'$; secondly, an estimate of the breadth of the earth's shadow where it crosses the moon: this he assumed to be 'the breadth of the moon' (Hypothesis 5).

The inaccuracy of the assumptions does not, however, detract from the mathematical interest of the succeeding inquiry. Here we find the logical sequence of propositions and the absolute rigour of demonstration characteristic of Greek mathematics. The only remaining drawback would be the practical impossibility of determining the exact moment when the moon is 'just halved'. The form and style of the book are classical, as befits the period between Euclid and Archimedes.

the method of view is no less interesting, for we have here the first specimen extant of pure geometry used with a *trigonometrical* object, in which respect it is a sort of forerunner of Archimedes's *Measurement of a Circle*. Aristarchus does not actually evaluate the trigonometrical ratios on which the ratios of the sizes and distances to be obtained depend; he finds limits between which they lie, and that by means of certain propositions which he assumes without proof, and which therefore must have been generally known to mathematicians of his day. These propositions are the equivalents of the statements that,

(1) if α is what we call the circular measure of an angle and α is less than $\frac{1}{2} \pi$, then the ratio $\sin \alpha / \alpha$ decreases, and the ratio $\tan \alpha / \alpha$ increases, as α increases from 0 to $\frac{1}{2} \pi$;

(2) if β be the circular measure of another angle less than $\frac{1}{2} \pi$, and $\alpha > \beta$, then

$$\frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}.$$

Aristarchus of course deals, not with actual circular measures, sines and tangents, but with angles (expressed not in degrees but as fractions of right angles), arcs of circles and their chords. Particular results obtained by Aristarchus are the equivalent of the following:

$$\frac{1}{18} > \sin 3^\circ > \frac{1}{20}, \quad [\text{Prop. 7}]$$

$$\frac{1}{45} > \sin 1^\circ > \frac{1}{60}, \quad [\text{Prop. 11}]$$

$$1 > \cos 1^\circ > \frac{89}{90}, \quad [\text{Prop. 12}]$$

$$1 > \cos^2 1^\circ > \frac{44}{45}. \quad [\text{Prop. 13}]$$

The book consists of eighteen propositions. Beginning with six hypotheses to the effect already indicated, Aristarchus declares that he is now in a position to prove

(1) that the distance of the sun from the earth is greater than eighteen times, but less than twenty times, the distance of the moon from the earth;

(2) that the diameter of the sun has the same ratio as aforesaid to the diameter of the moon;

(3) that the diameter of the sun has to the diameter of earth a ratio greater than 19:3, but less than 43:6.

The propositions containing these results are Props. 7 and 15.

Prop. 1 is preliminary, proving that two equal spheres comprehended by one cylinder, and two unequal spheres one cone with its vertex in the direction of the lesser sphere and the cylinder or cone touches the spheres in circles right angles to the line of centres. Prop. 2 proves that a sphere be illuminated by another sphere larger than itself the illuminated portion is greater than a hemisphere. Prop. 3 shows that the circle in the moon which divides the dark from the bright portion is least when the cone comprehending the sun and the moon has its vertex at our eye. The 'dividing circle', as we shall call it for short, which was in Hypothesis 6 spoken of as a great circle, is proved in Prop. 4 to be, not a great circle, but a small circle not perceptibly different from a great circle. The proof is typical and is worth giving along with that of some connected propositions (11 and 12).

B is the centre of the moon, A that of the earth, CD the diameter of the 'dividing circle in the moon', EF the parallel diameter in the moon. BA meets the circular section of the moon through A and EF in G , and CD in L . GH , GL are arcs each of which is equal to half the arc CE . By Hypothesis 6 the angle CAD is 'one-fifteenth of a sign' = 2° and the angle $BAC = 1^\circ$.

Now, says Aristarchus,

$$1^\circ : 45^\circ [> \tan 1^\circ : \tan 45^\circ]$$

$$> BC : CA,$$

and, *a fortiori*,

$$BC : BA \text{ or } BG : BA$$

$$< 1 : 45;$$

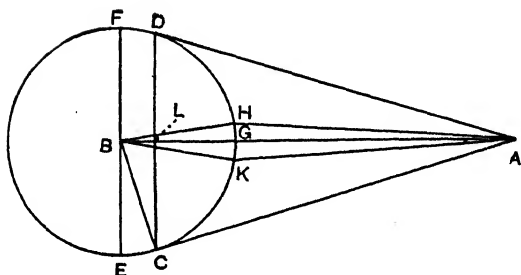
that is,

$$BG < \frac{1}{45} BA$$

$$> \angle HAB : \angle HBA,$$

whence

$$\angle HAB < \frac{1}{44} \angle HBA,$$



and (taking the doubles) $\angle HAK < \frac{1}{44} \angle HBK$.

But $\angle HBK = \angle EBC = \frac{1}{90} R$ (where R is a right angle);
therefore $\angle HAK < \frac{1}{3960} R$.

But 'a magnitude (arc HK) seen under such an angle is imperceptible to our eye';
therefore, *a fortiori*, the arcs CE , DF are severally imperceptible to our eye. Q. E. D.

An easy deduction from the same figure is Prop. 12, which shows that the ratio of CD , the diameter of the 'dividing circle', to EF , the diameter of the moon, is < 1 but $> \frac{89}{90}$.

We have $\angle EBC = \angle BAC = 1^\circ$;
therefore $(\text{arc } EC) = \frac{1}{90} (\text{arc } EG)$,
and accordingly $(\text{arc } CG) : (\text{arc } GE) = 89 : 90$.

Doubling the arcs, we have

$$(\text{arc } CGD) : (\text{arc } EGF) = 89 : 90.$$

But $CD : EF > (\text{arc } CGD) : (\text{arc } EGF)$
[equivalent to $\sin \alpha / \sin \beta > \alpha / \beta$, where $\angle CBD = 2\alpha$,
and $2\beta = \pi$];

therefore $CD : EF [= \cos 1^\circ] > 89 : 90$,

while obviously $CD : EF < 1$.

Prop. 11 finds limits to the ratio $EF : BA$ (the ratio of the diameter of the moon to the distance of its centre from the centre of the earth); the ratio is $< 2 : 45$ but $> 1 : 30$.

for

$$BC:BA < 1:45,$$

$$EF = 2 BC.$$

The second part requires the use of the circle drawn centre A and radius AC . This circle is that on which ends of the diameter of the 'dividing circle' move as the moves in a circle about the earth. If r is the radius of this circle, a chord in it equal to r subtends at the centre A an angle of $\frac{2}{3}R$ or 60° ; and the arc CD subtends an angle of 2° .

But (arc subtended by CD): (arc subtended by r)

$$< CD:r,$$

or

$$2:60 < CD:r;$$

that is,

$$CD:CA > 1:30.$$

And, by similar triangles,

$$CL:CA = CB:BA, \text{ or } CD:CA = 2 CB:BA = FE:BA$$

Therefore

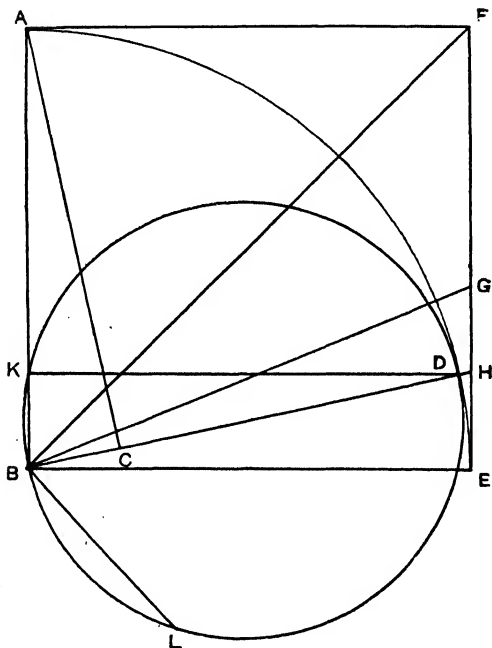
$$FE:BA > 1:30.$$

The proposition which is of the greatest interest on whole is Prop. 7, to the effect that *the distance of the sun from the earth is greater than 18 times, but less than 20 times, the distance of the moon from the earth*. This represents a great improvement on all previous attempts to estimate the relative distances. The first speculation on the subject was that of Anaximander (circa 611–545 B.C.), who seems to have made the distances of the sun and moon from the earth to be in the ratio 3:2. Eudoxus, according to Archimedes, made the diameter of the sun 9 times that of the moon, and Phidias, Archimedes's father, 12 times; and assuming that the angular diameters of the two bodies are equal, the ratio of their distances would be the same.

Aristarchus's proof is shortly as follows. A is the centre of the sun, B that of the earth, and C that of the moon at the moment of dichotomy, so that the angle ACB is right. ABE is a square, and AE is a quadrant of the sun's circular orbit. Join BF , and bisect the angle FBE by BG , so that

$$\angle GBE = \frac{1}{4}R \text{ or } 22\frac{1}{2}^\circ.$$

I. Now, by Hypothesis 4, $\angle ABC = 87^\circ$,
 so that $\angle HBE = \angle BAC = 3^\circ$;
 therefore $\angle GBE : \angle HBE = \frac{1}{4} R : \frac{1}{30} R$
 $= 15 : 2$,



so that $GE:HE [= \tan GBE : \tan HBE] > \angle GBE : \angle HBE$
 $> 15 : 2.$ (1)

The ratio which has to be proved $> 18:1$ is $AB:BC$ or $FE:EH$.

Now $FG:GE = FB:BE$,
 whence $FG^2:GE^2 = FB^2:BE^2 = 2:1$,

Therefore $FE:EG > 12:5$ or $36:15$.

Compounding this with (1) above, we have

$$FE:EH > 36:2 \text{ or } 18:1.$$

II. To prove $BA < 20 BC$.

Let BH meet the circle AE in D , and draw DK parallel to EB . Circumscribe a circle about the triangle BKD , and let the chord BL be equal to the radius (r) of the circle.

$$\text{Now} \quad \angle BDK = \angle DBE = \frac{1}{30} R,$$

so that arc $BK = \frac{1}{30}$ (circumference of circle).

$$\begin{aligned} \text{Thus} \quad (\text{arc } BK):(\text{arc } BL) &= \frac{1}{30} : \frac{1}{6}, \\ &= 1:10. \end{aligned}$$

$$\text{And} \quad (\text{arc } BK):(\text{arc } BL) < BK:r$$

[this is equivalent to $\alpha/\beta < \sin \alpha/\sin \beta$, where $\alpha < \beta < \frac{1}{2}\pi$],

$$\text{so that} \quad r < 10 BK,$$

$$\text{and} \quad BD < 20 BK.$$

$$\text{But} \quad BD:BK = AB:BC;$$

$$\text{therefore} \quad AB < 20 BC.$$

Q. E. D.

The remaining results obtained in the treatise can be visualized by means of the three figures annexed, which have reference to the positions of the sun (centre A), the earth (centre B) and the moon (centre C) during an eclipse. Fig. 1 shows the middle position of the moon relatively to the earth's shadow which is bounded by the cone comprehending the sun and the earth. ON is the arc with centre B along which move the extremities of the diameter of the dividing circle in the moon. Fig. 3 shows the same position of the moon in the middle of the shadow, but on a larger scale. Fig. 2 shows the moon at the moment when it has just entered the shadow; and, as the breadth of the earth's shadow is that of two moons (Hypothesis 5), the moon in the position shown in Fig. 2

while Y, Z are the points in which the perpendicular through A , the centre of the sun, to BA meets the cone enveloping the sun and the earth.

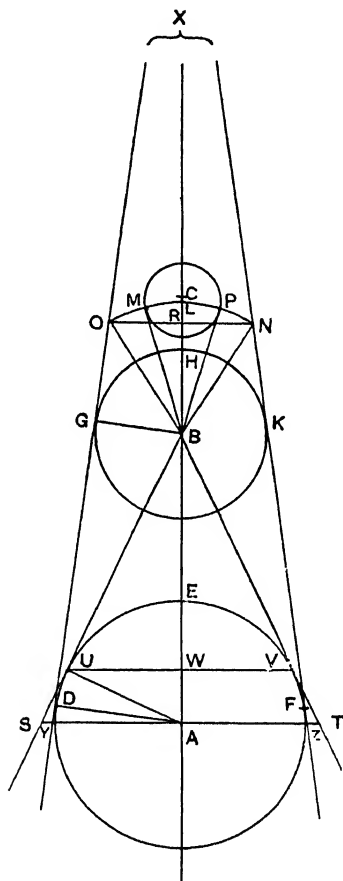


FIG. 1.

This being premised, the main results obtained are as follows:

(2)

$$ON : (\text{diam. of sun}) < 1 : 9$$

but

$$> 22 : 225.$$

(3)

$$ON : YZ > 979 : 10125.$$

Prop. 14 (Fig. 3).

Prop. 15.

$$BC : CS > 675 : 1.$$

$$(\text{Diam. of sun}) : (\text{diam. of earth}) > 19 : 3$$

but

$$< 43 : 6.$$

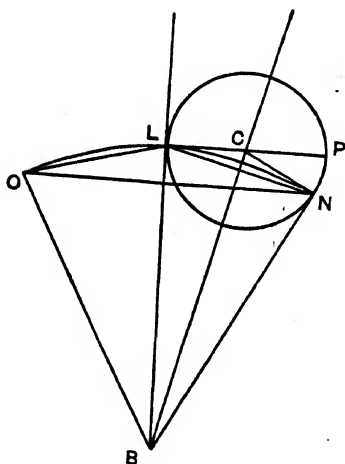


FIG. 2.

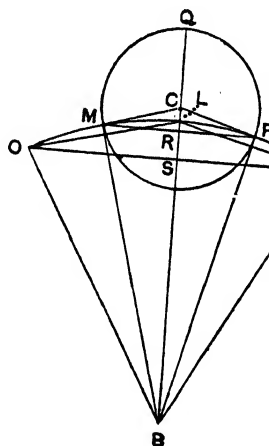


FIG. 3.

Prop. 17.

$$(\text{Diam. of earth}) : (\text{diam. of moon}) > 108 : 43$$

but

$$< 60 : 19.$$

It is worth while to show how these results are proved

Prop. 13.

(1) In Fig. 2 it is clear that

$$ON < 2LN \text{ and, a fortiori, } < 2LP.$$

The triangles LON , CLN being

Hence $ON:LC = ON^2:NL^2$

$$> 89^2:45^2;$$

therefore $ON:LP > 7921:4050$

$$> 88:45, \text{ says Aristarchus.}$$

[If $\frac{7921}{4050}$ be developed as a continued fraction, we easily obtain $1 + \frac{1}{1 + \frac{1}{21 + \frac{1}{2}}}$, which is in fact $\frac{88}{45}$.]

(2) $ON < 2$ (diam. of moon).

But (diam. of moon) $< \frac{1}{18}$ (diam. of sun); (Prop. 7)

therefore $ON < \frac{1}{9}$ (diam. of sun).

Again $ON:(\text{diam. of moon}) > 88:45$, from above,

and (diam. of moon):(diam. of sun) $> 1:20$; (Prop. 7)

therefore, *ex aequali*,

$$ON:(\text{diam. of sun}) > 88:900$$

$$> 22:225.$$

(3) Since the same cone comprehends the sun and the moon, the triangle BUV (Fig. 1) and the triangle BLN (Fig. 2) are similar, and

$$LN:LP = UV:(\text{diam. of sun})$$

$$= WU:UA$$

$$= UA:AS$$

$$< UA:AY.$$

But $LN:LP > 89:90$; (Prop. 12)

therefore, *a fortiori*, $UA:AY > 89:90$.

And $UA:AY = 2UA:YZ$

$$= (\text{diam. of sun}):YZ.$$

But $ON:(\text{diam. of sun}) > 22:225$; (Prop. 12)

Prop. 14 (Fig. 3).

The arcs OM , ML , LP , PN are all equal; therefore the chords. BM , BP are tangents to the circle MQI . CM is perpendicular to BM , while BM is perpendicular to CM . Therefore the triangles LOS , CMR are similar.

Therefore $SO : MR = SL : RC$.

But $SO < 2 MR$, since $ON < 2 MP$; (

therefore $SL < 2 RC$,

and, *a fortiori*, $SR < 2 RC$, or $SC < 3 RC$,

that is, $CR : CS > 1 : 3$.

Again, $MC : CR = BC : CM$

$> 45 : 1$; (see I

therefore, *ex aequali*,

$CM : CS > 15 : 1$.

And $BC : CM > 45 : 1$;

therefore $BC : CS > 675 : 1$.

Prop. 15 (Fig. 1).

We have $NO : (\text{diam. of sun}) < 1 : 9$, (I

and, *a fortiori*, $YZ : NO > 9 : 1$;

therefore, by similar triangles, if YO , ZN meet in X ,

$AX : XR > 9 : 1$,

and *convertendo*, $XA : AR < 9 : 8$.

But $AB > 18 BC$, and, *a fortiori*, $> 18 BR$,

whence $AB > 18 (AR - AB)$, or $19 AB > 18 AR$;

that is, $AR : AB < 19 : 18$.

Therefore, *ex aequali*,

$XA : AB < 19 : 16$,

and, *convertendo*, $AX : XB > 19 : 3$;

But $AB:BC < 20:1$;

Therefore, *ex aequali*,

$$AB:BR < 13500:674 \\ < 6750:337,$$

Hence, by inversion and *componendo*,

$$RA:AB > 7087:6750. \quad (1)$$

But $AX:XR = YZ:NO$

$$< 10125:979; \quad (\text{Prop. 13})$$

Therefore, *convertendo*,

$$XA:AR > 10125:9146.$$

From this and (1) we have, *ex aequali*,

$$XA:AB > 10125 \times 7087 : 9146 \times 6750 \\ > 71755875 : 61735500 \\ > 43:37, \text{ a fortiori.}$$

[It is difficult not to see in 43:37 the expression $1 + \frac{1}{6} + \frac{1}{6}$, which suggests that 43:37 was obtained by developing the ratio as a continued fraction.]

Therefore, *convertendo*,

$$XA:XB < 43:6,$$

Hence (diam. of sun) : (diam. of earth) < 43:6. Q. E. D.

XIII

ARCHIMEDES

THE siege and capture of Syracuse by Marcellus during the second Punic war furnished the occasion for the appearance of Archimedes as a personage in history; it is with this event that most of the detailed stories of him are connected; and the fact that he was killed in the sack of the city in 212 B.C., when he is supposed to have been 75 years of age, enables us to fix his date at about 287-212 B.C. His father was Phidias, the astronomer, and was on intimate acquaintance with, if not related to, King Hieron and his son Gelon. It appears from a passage of Diodorus that he spent some time in Egypt, which visit was the occasion of his discovery of the so-called Archimedean screw as a means of pumping water. It may be inferred that he studied at Alexandria under the successors of Euclid. It was probably at Alexandria that he made the acquaintance of Conon of Samos (for whom he had the highest regard both as a mathematician and a friend) and of Eratosthenes of Cyrene. To the former he was indebted for the opportunity of communicating his discourses before their publication, while it was to Eratosthenes that he sent *The Method* with an introductory letter which is of the highest interest, (if we may judge by its heading) the famous Cattle-

Traditions.

It is natural that history or legend should say more of his mechanical inventions than of his mathematical achievements, which would appeal less to the average mind. His inventions were used with great effect against the Romans in the

short range, machines for discharging showers of missiles through holes made in the walls, and others consisting of long movable poles projecting beyond the walls which either dropped heavy weights on the enemy's ships, or grappled their prows by means of an iron hand or a beak like that of a crane, then lifted them into the air and let them fall again.¹ Marcellus is said to have derided his own engineers with the words, 'Shall we not make an end of fighting against this geometrical Briareus who uses our ships like cups to ladle water from the sea, drives off our *sambuca* ignominiously with cudgel-blows, and by the multitude of missiles that he hurls at us all at once outdoes the hundred-handed giants of mythology?'; but all to no purpose, for the Romans were in such abject terror that, 'if they did but see a piece of rope or wood projecting above the wall, they would cry "there it is", declaring that Archimedes was setting some engine in motion against them, and would turn their backs and run away'.² These things, however, were merely the 'diversions of geometry at play',³ and Archimedes himself attached no importance to them. According to Plutarch,

'though these inventions had obtained for him the renown of more than human sagacity, he yet would not even deign to leave behind him any written work on such subjects, but, regarding as ignoble and sordid the business of mechanics and every sort of art which is directed to use and profit, he placed his whole ambition in those speculations the beauty and subtlety of which is untainted by any admixture of the common needs of life.'⁴

(a) *Astronomy.*

Archimedes did indeed write one mechanical book, *On Sphere-making*, which is lost; this described the construction of a sphere to imitate the motions of the sun, moon and planets.⁵ Cicero saw this contrivance and gives a description of it; he says that it represented the periods of the moon and the apparent motion of the sun with such accuracy that

the making of spheres and produce a model of the heavens by the means of the circular motion of water', it is possible that Archimedes's sphere was moved by water. In any case Archimedes was much occupied with astronomy. Livy calls him 'unicus spectator caeli siderumque'.¹ Hipparchus says that from these observations it is clear that the differences in the observations are altogether small, but, as to the solstices, I almost always find that Archimedes and I have both erred to the extent of a quarter of a day both in the observation and in the deduction therefrom'.² Archimedes then had evidently considered the length of the year. Macrobius says he discovered the positions of the planets,³ and he himself describes in his *rechner* the apparatus by which he measured the angular diameter of the sun.

(β) *Mechanics.*

Archimedes wrote, as we shall see, on theoretical mechanics and it was by theory that he solved the problem *To move a given weight by a given force*, for it was in reliance on the irresistible cogency of his proof that he declared to Hieron that any given weight could be moved by any given force (however small), and boasted that, 'if he were given a place to stand on, he could move the earth' ($\pi\alpha\ \beta\omega, \text{ καὶ κινῶ τὴν γῆν}$ as he said in his Doric dialect). The story, told by Plutarch, is that, 'when Hieron was struck with amazement and curiosity, he ordered Archimedes to reduce the problem to practice and to give an illustration of some great weight moved by a small force'. The weight was fixed upon a ship of burden with three masts from the arsenal which had only been drawn up with great labour by many men, and loading her with many passengers and freight, himself the while sitting far off, with no great exertion but only holding the end of a compound pulley (πολύσπινδος) quietly in his hand and pulling at it, he drew the ship smoothly and safely as if she were moving through the water.

The story that Archimedes set the Roman ships on fire by an arrangement of burning-glasses or concave mirrors is not found in any authority earlier than the 17th century.

likely that he discovered some form of burning-mirror, e.g. a paraboloid of revolution, which would reflect to one point all rays falling on its concave surface in a direction parallel to its axis.

Archimedes's own view of the relative importance of his many discoveries is well shown by his request to his friends and relatives that they should place upon his tomb a representation of a cylinder circumscribing a sphere, with an inscription giving the ratio which the cylinder bears to the sphere; from which we may infer that he regarded the discovery of this ratio as his greatest achievement. Cicero, when quaestor in Sicily, found the tomb in a neglected state and repaired it¹; but it has now disappeared, and no one knows where he was buried.

Archimedes's entire preoccupation by his abstract studies is illustrated by a number of stories. We are told that he would forget all about his food and such necessities of life, and would be drawing geometrical figures in the ashes of the fire or, when anointing himself, in the oil on his body.² Of the same sort is the tale that, when he discovered in a bath the solution of the question referred to him by Hieron, as to whether a certain crown supposed to have been made of gold did not in fact contain a certain proportion of silver, he ran naked through the street to his home shouting *εὕρηκα, εὕρηκα*.³ He was killed in the sack of Syracuse by a Roman soldier. The story is told in various forms; the most picturesque is that found in Tzetzes, which represents him as saying to a Roman soldier who found him intent on some diagrams which he had drawn in the dust and came too close, 'Stand away, fellow, from my diagram', whereat the man was so enraged that he killed him.⁴

Summary of main achievements.

In geometry Archimedes's work consists in the main of original investigations into the quadrature of curvilinear plane figures and the quadrature and cubature of curved

birth to the calculus of the infinite conceived and brought to perfection successively by Kepler, Cavalieri, Fermat and Newton'. He performed in fact what is equivalent to *integration* in finding the area of a parabolic segment, a spiral, the surface and volume of a sphere and a segment of a sphere, and the volumes of any segments of the sphere by revolution of the second degree. In arithmetic he gave approximations to the value of π , in the course of which calculation he shows that he could approximate to the square roots of large or small non-square numbers; he invented a system of arithmetical terminology by which he could express in language any number up to that which should write down with 1 followed by 80,000 million ciphers. In mechanics he not only worked out the principles of the subject but advanced so far as to find the centre of gravity of a segment of a parabola, a semicircle, a cone, a hemisphere, a segment of a sphere, a right segment of a paraboloid, a spheroid of revolution. His mechanics, as we shall see, become more important in relation to his geometry than the discovery of the treatise called *The Method* which was supposed to be lost. Lastly, he invented the whole science of hydrostatics, which again he carried so far as to give a complete investigation of the positions of rest and stability of a right segment of a paraboloid of revolution floated in a fluid with its base either upwards or downwards, but in which the base is either wholly above or wholly below the surface of the fluid. This represents a sum of mathematical knowledge unsurpassed by any one man in the world's history.

Character of treatises.

The treatises are, without exception, monuments of mathematical exposition; the gradual revelation of the subject, the attack, the masterly ordering of the propositions, the elimination of everything not immediately relevant to the purpose, the finish of the whole, are so impressive in their perfection as to create a feeling akin to awe in the mind of the reader. As Plutarch said 'It is not possible to

time a certain mystery veiling the way in which he reached at his results. For it is clear that they were not covered by the steps which lead up to them in the finished treatises. If the geometrical treatises stood alone, Archimedes might seem, as Wallis said, 'as it were of set purpose have covered up the traces of his investigation, as if he had wished posterity the secret of his method of inquiry, while he wished to extort from them assent to his results'. And indeed (again in the words of Wallis) 'not only Archimedes hid nearly all the ancients so hid from posterity their method of analysis (though it is clear that they had one) that more modern mathematicians found it easier to invent a new analysis than to seek out the old'. A partial exception is furnished by *The Method* of Archimedes, so happily discovered by Heiberg. In this book Archimedes tells us how he discovered certain theorems in quadrature and cubature, not by the use of mechanics, weighing elements of a figure against elements of another simpler figure the mensuration of which was already known. At the same time he is careful to insist on the difference between (1) the means which may be sufficient to suggest the truth of theorems, though not furnishing scientific proofs of them, and (2) the rigorous demonstrations of them by orthodox geometrical methods which must follow before they can be finally accepted and published:

'In such things', he says, 'first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the aid, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge.' 'This', he adds, 'is a reason why, in the case of the theorems that the volumes of a cone and a pyramid are one-third of the volumes of the cylinder and prism respectively having the same base and equal height, the proofs of which Eudoxus was first to discover: no small share of the credit should be

on the *Quadrature of the parabola*, namely that *the area of any segment of a section of a right-angled cone (i. e. a parabola) is four-thirds of that of the triangle which has the same base and height*. The mechanical proof, however, of this theorem in the *Quadrature of the Parabola* is different from that in the *Method*, and is more complete in that the argument is clinched by formally applying the method of exhaustion.

List of works still extant.

The extant works of Archimedes in the order in which they appear in Heiberg's second edition, following the order of the manuscripts so far as the first seven treatises are concerned, are as follows :

- (5) *On the Sphere and Cylinder* : two Books.
- (9) *Measurement of a Circle*.
- (7) *On Conoids and Spheroids*.
- (6) *On Spirals*.
- (1) *On Plane Equilibriums*, Book I.
- (3) " " " Book II.
- (10) *The Sand-reckoner (Psummites)*.
- (2) *Quadrature of the Parabola*.
- (8) *On Floating Bodies* : two Books.
- ? *Stomachion* (a fragment).
- (4) *The Method*.

This, however, was not the order of composition ; and, judging (*a*) by statements in Archimedes's own prefaces to certain of the treatises and (*b*) by the use in certain treatises of results obtained in others, we can make out an approximate chronological order, which I have indicated in the above list by figures in brackets. The treatise *On Floating Bodies* was formerly only known in the Latin translation by William of Moerbeke, but the Greek text of it has now been in great part restored by Heiberg from the Constantinople manuscript which also contains *The Method* and the fragment of the *Stomachion*.

Besides these works we have a collection of propositions (*Liber assumptorum*) which has reached us through the

QURRA the book is attributed to Archimedes, the propositions cannot be his in their present form, since his name is several times mentioned in them; but it is quite likely that some of them are of Archimedean origin, notably those about the geometrical figures called $\alpha\rho\beta\eta\lambda\omicron\varsigma$ ('shoemaker's knife') and $\sigma\acute{\alpha}\lambda\iota\nu\omicron\nu$ (probably 'salt-cellar') respectively and Prop. 8 bearing on the trisection of an angle.

There is also the *Cattle-Problem* in epigrammatic form, which purports by its heading to have been communicated by Archimedes to the mathematicians at Alexandria in a letter to Eratosthenes. Whether the epigrammatic form is due to Archimedes himself or not, there is no sufficient reason for doubting the possibility that the substance of it was set as a problem by Archimedes.

Traces of lost works.

Of works which are lost we have the following traces.

1. Investigations relating to *polyhedra* are referred to by Pappus who, after alluding to the five regular polyhedra, describes thirteen others discovered by Archimedes which are semi-regular, being contained by polygons equilateral and equiangular but not all similar.¹

2. There was a book of arithmetical content dedicated to Zeuxippus. We learn from Archimedes himself that it dealt with the *naming of numbers* ($\kappa\alpha\tau\omicron\nu\acute{o}\mu\alpha\chi\iota\varsigma\ \tau\hat{\omega}\nu\ \acute{\alpha}\rho\iota\theta\mu\hat{\omega}\nu$)² and expounded the system, which we find in the *Sand-reckoner*, of expressing numbers higher than those which could be written in the ordinary Greek notation, numbers in fact (as we have said) up to the enormous figure represented by 1 followed by 80,000 million million ciphers.

3. One or more works on mechanics are alluded to containing propositions not included in the extant treatise *On Plane Equilibriums*. Pappus mentions a work *On Balances* or *Levers* ($\pi\epsilon\rho\iota\ \zeta\upsilon\gamma\hat{\omega}\nu$) in which it was proved (as it also was in Philon's and Heron's *Mechanics*) that 'greater circles overpower lesser circles when they revolve about the same centre'.³ Heron, too, speaks of writings of Archimedes 'which bear the title of

“works on the lever”¹.¹ Simplicius refers to *problems on the centre of gravity*, *κεντροβαρικά*, such as the many elegant problems solved by Archimedes and others, the object of which is to show how to find the centre of gravity, that is, the point in a body such that if the body is hung up from it, the body will remain at rest in any position.² This recalls the assumption in the *Quadrature of the Parabola* (6) that, if a body hangs at rest from a point, the centre of gravity of the body and the point of suspension are in the same vertical line. Pappus has a similar remark with reference to a point of *support*, adding that the centre of gravity is determined as the intersection of two straight lines in the body, through two points of support, which straight lines are vertical when the body is in equilibrium so supported. Pappus also gives the characteristic of the centre of gravity mentioned by Simplicius, observing that this is the most fundamental principle of the theory of the centre of gravity, the elementary propositions of which are found in Archimedes’s *On Equilibriums* (*περὶ ἰσορροπιῶν*) and Heron’s *Mechanics*. Archimedes himself cites propositions which must have been proved elsewhere, e.g. that the centre of gravity of a cone divides the axis in the ratio 3 : 1, the longer segment being that adjacent to the vertex³; he also says that ‘it is proved in the *Equilibriums*’ that the centre of gravity of any segment of a right-angled conoid (i.e. paraboloid of revolution) divides the axis in such a way that the portion towards the vertex is double of the remainder.⁴ It is possible that there was originally a larger work by Archimedes *On Equilibriums* of which the surviving books *On Plane Equilibriums* formed only a part; in that case *περὶ ζυγῶν* and *κεντροβαρικά* may only be alternative titles. Finally, Heron says that Archimedes laid down a certain procedure in a book bearing the title ‘Book on Supports’.⁵

4. Theon of Alexandria quotes a proposition from a work of Archimedes called *Catoptrica* (properties of mirrors) to the effect that things thrown into water look larger and still larger the farther they sink.⁶ Olympiodorus, too, mentions

¹ Heron, *Mechanics*, i. 32.

² Simpl. on Arist. *De caelo*, ii, p. 508 a 30, Brandis; p. 543. 24, Heib.

³ *Method*, Lemma 10.

⁴ *On Floating Bodies*, ii. 2.

⁵ Heron, *Mechanics*, i. 25.

that Archimedes proved the phenomenon of refraction 'by means of the ring placed in the vessel (of water)'.¹ A scholiast to the Pseudo-Euclid's *Catoptrica* quotes a proof, which he attributes to Archimedes, of the equality of the angles of incidence and of reflection in a mirror.

The text of Archimedes.

Heron, Pappus and Theon all cite works of Archimedes which no longer survive, a fact which shows that such works were still extant at Alexandria as late as the third and fourth centuries A.D. But it is evident that attention came to be concentrated on two works only, the *Measurement of a Circle* and *On the Sphere and Cylinder*. Eutocius (fl. about A.D. 500) only wrote commentaries on these works and on the *Plane Equilibriums*, and he does not seem even to have been acquainted with the *Quadrature of the Parabola* or the work *On Spirals*, although these have survived. Isidorus of Miletus revised the commentaries of Eutocius on the *Measurement of a Circle* and the two Books *On the Sphere and Cylinder*, and it would seem to have been in the school of Isidorus that these treatises were turned from their original Doric into the ordinary language, with alterations designed to make them more intelligible to elementary pupils. But neither in Isidorus's time nor earlier was there any collected edition of Archimedes's works, so that those which were less read tended to disappear.

In the ninth century Leon, who restored the University of Constantinople, collected together all the works that he could find at Constantinople, and had the manuscript written (the archetype, Heiberg's A) which, through its derivatives, was, up to the discovery of the Constantinople manuscript (C) containing *The Method*, the only source for the Greek text. Leon's manuscript came, in the twelfth century, to the Norman Court at Palermo, and thence passed to the House of Hohenstaufen. Then, with all the library of Manfred, it was given to the Pope by Charles of Anjou after the battle of Benevento in 1266. It was in the Papal Library in the years 1269 and 1271, but some time after 1269, which

“works on the lever”¹. Simplicius refers to *problem centre of gravity*, *κεντροβαρική*, such as the many problems solved by Archimedes and others, the object of is to show how to find the centre of gravity, that is, the point in a body such that if the body is hung up from it, it will remain at rest in any position.² This recalls the proposition in the *Quadrature of the Parabola* (6) that, if a body is suspended at rest from a point, the centre of gravity of the body and the point of suspension are in the same vertical line. Pappus makes a similar remark with reference to a point of *support*, saying that the centre of gravity is determined as the intersection of two straight lines in the body, through two points of support, which straight lines are vertical when the body is in equilibrium so supported. Pappus also gives the characteristic of the centre of gravity mentioned by Simplicius, observing that the most fundamental principle of the theory of the centre of gravity, the elementary propositions of which are found in Archimedes’s *On Equilibriums* (*περὶ ἰσορροπιῶν*) and *Mechanics*. Archimedes himself cites propositions which have been proved elsewhere, e.g. that the centre of gravity of a cone divides the axis in the ratio 3 : 1, the longer segment being that adjacent to the vertex³; he also says that he proved in the *Equilibriums* that the centre of gravity of a segment of a right-angled conoid (i.e. paraboloid of revolution) divides the axis in such a way that the portion towards the vertex is double of the remainder.⁴ It is possible that this was originally a larger work by Archimedes *On Equilibriums* of which the surviving books *On Plane Equilibrium* and *On Spirals* are only a part; in that case *περὶ ζυγῶν* and *κεντροβαρική* may be only alternative titles. Finally, Heron says that Archimedes laid down a certain procedure in a book bearing the title ‘Book on Supports’.⁵

4. Theon of Alexandria quotes a proposition from a work of Archimedes called *Catoptrica* (properties of mirrors) to the effect that things thrown into water look larger when they are larger the farther they sink.⁶ Olympiodorus, too,

Archimedes proved the phenomenon of refraction 'by the ring placed in the vessel (of water)'.¹ A scholiast Pseudo-Euclid's *Catoptrica* quotes a proof, which he ascribes to Archimedes, of the equality of the angles of incidence and of reflection in a mirror.

The text of Archimedes.

Apollonius, Pappus and Theon all cite works of Archimedes no longer survive, a fact which shows that such works were no longer extant at Alexandria as late as the third and fourth centuries A.D. But it is evident that attention came to be concentrated on two works only, the *Measurement of a Circle* and the *Sphere and Cylinder*. Eutocius (fl. about A.D. 500) wrote commentaries on these works and on the *Plane Numbers*, and he does not seem even to have been acquainted with the *Quadrature of the Parabola* or the work on *Conoids and Spheroids*, although these have survived. Isidorus of Miletus wrote the commentaries of Eutocius on the *Measurement of a Circle* and the two Books *On the Sphere and Cylinder*, and would seem to have been in the school of Isidorus. These treatises were turned from their original Doric or ordinary language, with alterations designed to make them more intelligible to elementary pupils. But neither in Isidorus's time nor earlier was there any collected edition of Archimedes's works, so that those which were less read tended to disappear.

In the ninth century Leon, who restored the University at Constantinople, collected together all the works that he could find at Constantinople, and had the manuscript written in a new type, Heiberg's A) which, through its derivatives, led to the discovery of the Constantinople manuscript (C) giving *The Method*, the only source for the Greek text. This manuscript came, in the twelfth century, to the Norman Court at Palermo, and thence passed to the House of Hohenstaufen. Then, with all the library of Manfred, it

private hands. In 1491 it belonged to Georgius Valla, who translated from it the portions published in his posthumous work *De expetendis et fugiendis rebus* (1501), and intended to publish the whole of Archimedes with Eutocius's commentaries. On Valla's death in 1500 it was bought by Albertus Pius, Prince of Carpi, passing in 1530 to his nephew, Rodolphus Pius, in whose possession it remained till 1544. At some time between 1544 and 1564 it disappeared, leaving no trace.

The greater part of A was translated into Latin in 1229 by William of Moerbeke at the Papal Court at Viterbo. This translation, in William's own hand, exists at Rome (Cotton. lat. 1850, Heiberg's B), and is one of our principal sources, for, although the translation was hastily done and the translator sometimes misunderstood the Greek, he followed its wording so closely that his version is, for purposes of collation, as good as a Greek manuscript. William used also for his translation, another manuscript from the same library which contained works not included in A. This manuscript was a collection of works on mechanics and optics; William translated from it the two Books *On Floating Bodies*, and also contained the *Plane Equilibriums* and the *Quadrature of the Parabola*, for which books William used both manuscripts.

The four most important extant Greek manuscripts (except C, the Constantinople manuscript discovered in 1906) were copied from A. The earliest is E, the Venice manuscript (Marcianus 305), which was written between the years 1400 and 1472. The next is D, the Florence manuscript (Laurenzianus XXVIII. 4), which was copied in 1491 for Angelo Poliziano, permission having been obtained with some difficulty in consequence of the jealousy with which Valla guarded his treasure. The other two are G (Paris. 2360) copied from A after it passed to Albertus Pius, and H (Paris. 2361) copied in 1530 by Christopherus Auverus for Georges d'Armagnac, Bishop of Rodez. These four manuscripts, with the translation

Nicholas though it does not seem to have belonged to him. Regiomontanus made a copy of this translation about 1468 and revised it with the help of E (the Venice manuscript of the Greek text) and a copy of the same translation belonging to Cardinal Bessarion, as well as another 'old copy' which seems to have been B.

The *editio princeps* was published at Basel (*apud Hervagium*) by Thomas Gechauff Venatorius in 1544. The Greek text was based on a Nürnberg MS. (Nürnberg. Cent. V, app. 12) which was copied in the sixteenth century from A but with interpolations derived from B; the Latin translation was Regiomontanus's revision of Jacobus Cremonensis (Nürnberg. Cent. V, 15).

A translation by F. Commandinus published at Venice in 1558 contained the *Measurement of a Circle, On Spirals, the Quadrature of the Parabola, On Conoids and Spheroids*, and the *Sand-reckoner*. This translation was based on the Basel edition, but Commandinus also consulted E and other Greek manuscripts.

Torelli's edition (Oxford, 1792) also followed the *editio princeps* in the main, but Torelli also collated E. The book was brought out after Torelli's death by Abram Robertson, who also collated five more manuscripts, including D, G and H. The collation, however, was not well done, and the edition was not properly corrected when in the press.

The second edition of Heiberg's text of all the works of Archimedes with Eutocius's commentaries, Latin translation, apparatus criticus, &c., is now available (1910-15) and, of course, supersedes the first edition (1880-1) and all others. It naturally includes *The Method*, the fragment of the *Stomachion*, and so much of the Greek text of the two Books *On Floating Bodies* as could be restored from the newly discovered Constantinople manuscript.¹

Contents of *The Method*.

Our description of the extant works of Archimedes may suitably begin with *The Method* (the full title is *On*

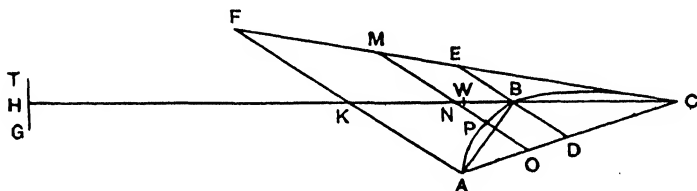
¹ *The Works of Archimedes*, edited in modern notation by the present writer in 1897, was based on Heiberg's first edition, and the Supplement

Mechanical Theorems, Method (communicated) to Premising certain propositions in mechanics from the *Plane Equilibriums*, and a lemma Prop. 1 of *On Conoids and Spheroids*, Archimedes applies his mechanical method the following results. The volume of a segment of a section of a right-angled cone (parallelepiped) is equal to the triangle with the same base and height (Prop. 2). The volume of a right cylinder circumscribing a sphere or a spheroid is equal to the triangle with axis equal to the diameter or axis of the sphere or spheroid is $1\frac{1}{2}$ times the sphere or spheroid respectively (Props. 2, 3). Props. 4, 7, 8, 11 find the volume of any segment cut off, by a plane at right angles to the axis, from any right-angled conoid (paraboloid of revolution, sphere, spheroid, and obtuse-angled conoid) in terms of the cone which has the same base as the segment and equal height. In Props. 5, 6, 9, 10 Archimedes uses the method to find the centre of gravity of a segment of a cone of revolution, a sphere, and a spheroid respectively. Props. 12-15 and Prop. 16 are concerned with the centres of gravity of special solid figures. (1) Suppose a prism with a square base to have a cylinder inscribed in it, the circular bases of the cylinder being circles inscribed in the squares of the bases of the prism, and suppose a plane drawn through one side of one base of the prism and through the opposite side of the circle in the opposite base which is parallel to the other side. This plane cuts off a solid bounded by the cylinder and by part of the curved surface of the cylinder (like a hoof cut off by a plane); and Props. 12-15 find that its volume is one-sixth of the volume of the prism. (2) Suppose a cylinder inscribed in a cube, so that the bases of the cylinder are circles inscribed in two opposite faces of the cube, and suppose another cylinder similar to the first with reference to two other opposite faces. These two cylinders enclose a certain solid which is actually made of two 'hoofs' like that of Prop. 12. Prop. 16 proves that the volume of this solid is two-thirds of that of the cube. Archimedes observes in his preface that

solid enclosed by *planes*, whereas the volume of curvilinear solids (spheres, spheroids, &c.) is generally only expressible in terms of other curvilinear solids (cones and cylinders). In accordance with his dictum that the results obtained by the mechanical method are merely indicated, but not actually proved, unless confirmed by the rigorous methods of pure geometry, Archimedes proved the facts about the two last-named solids by the orthodox method of exhaustion as regularly used by him in his other geometrical treatises; the proofs, partly lost, were given in Props. 15 and 16.

We will first illustrate the method by giving the argument of Prop. 1 about the area of a parabolic segment.

Let ABC be the segment, BD its diameter, CF the tangent at C . Let P be any point on the segment, and let AKF ,



$OPNM$ be drawn parallel to BD . Join CB and produce it to meet MO in N and FA in K , and let KH be made equal to KC .

Now, by a proposition 'proved in a lemma' (cf. *Quadrature of the Parabola*, Prop. 5)

$$\begin{aligned} MO:OP &= CA:AO \\ &= CK:KN \\ &= HK:KN. \end{aligned}$$

Also, by the property of the parabola, $EB = BD$, so that $MN = NO$ and $FK = KA$.

It follows that, if HC be regarded as the bar of a balance, a line TG equal to PO and placed with its middle point at H balances, about K , the straight line MO placed where it is, i.e. with its middle point at N .

Similarly with *all* lines, as MO , PO , in the triangle CFA and the segment CBA respectively.

And there are the same number of these lines. Therefore

triangle CFA placed where it is.

But the centre of gravity of the triangle where $CW = 2 WK$ [and the whole triangle m acting at W].

$$\text{Therefore } (\text{segment } ABC) : \triangle CFA = WK : CW \\ = 1 : 3,$$

$$\text{so that } (\text{segment } ABC) = \frac{1}{3} \triangle CFA \\ = \frac{4}{3} \triangle ABC.$$

It will be observed that Archimedes takes the triangle to be *made up* of parallel lines in together. In reality they are made up of infinite strips, but the width (dx , as we might say) being the same for the elements of the triangle and segment divides out. And of course the weight of each both is proportional to the area. Archimedes mentioning *moments*, in effect assumes that the moments of each particle of a figure, acting at its centre of gravity, are equal to the moment of the whole figure acting at its centre of gravity.

We will now take the case of any segment of revolution, because that will cover several Archimedes as particular cases.

The ellipse with axes AA' , BB' is a section of the plane of the paper in a spheroid with axis AA' . To find the volume of any right segment ADC in terms of the right cone with the same base and height.

Let DC be the diameter of the circular base. Join AB , AB' , and produce them to meet the ellipse in K , K' , and DC produced in E , F .

Conceive a cylinder described with axis AA' and circle on KK' as diameter, and cones described with axis and bases the circles on EF , DC as diameters.

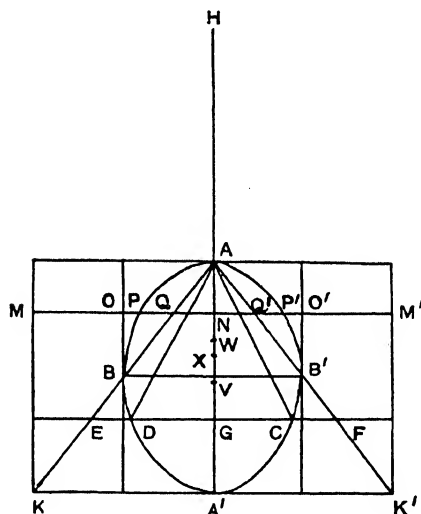
Let N be any point on AG , and let $MOPQ$ be a line drawn through N parallel to BB' or DC as diameter.

Produce $A'A$ to H so that $HA = AA'$.

Now

$$\begin{aligned} HA : AN &= A'A : AN \\ &= KA : AQ \\ &= MN : NQ \\ &= MN^2 : MN \cdot NQ. \end{aligned}$$

It is now necessary to prove that $MN \cdot NQ = NP^2 + NQ^2$.



By the property of the ellipse,

$$\begin{aligned} AN \cdot NA' : NP^2 &= (\tfrac{1}{2} AA')^2 : (\tfrac{1}{2} BB')^2 \\ &= AN^2 : NQ^2; \end{aligned}$$

therefore
$$NQ^2 : NP^2 = AN^2 : AN \cdot NA'$$

$$= NQ^2 : NQ \cdot QM,$$

whence
$$NP^2 = MQ \cdot QN.$$

Add NQ^2 to each side, and we have

$$NP^2 + NQ^2 = MN \cdot NQ.$$

angles to AA' in the cylinder, the spheroid and the cone respectively.

Therefore, if HAA' be a lever, and the sections spheroid and cone be both placed with their centres of gravity at H , these sections placed at H balance, about A , the axis MM' of the cylinder where it is.

Treating all the corresponding sections of the spheroid, the cone and the cylinder in the same manner, we find that the cylinder with axis AG , where it is, balances about A , the cone AEF and the segment ADC together, both are placed with their centres of gravity at H . If W be the centre of gravity of the cylinder, i.e. the point of AG ,

$$HA : AW = (\text{cylinder, axis } AG) : (\text{cone } AEF + \text{segmt. } ADC)$$

If we call V the volume of the cone AEF , and S the volume of the segment of the spheroid, we have

$$(\text{cylinder}) : (V + S) = 3V \cdot \frac{AA'^2}{AG^2} : (V + S),$$

while $HA : AW = AA' : \frac{1}{2} AG$.

Therefore $AA' : \frac{1}{2} AG = 3V \cdot \frac{AA'^2}{AG^2} : (V + S)$,

and $(V + S) = \frac{3}{2} V \cdot \frac{AA'}{AG}$,

whence $S = V \left(\frac{3AA'}{2AG} - 1 \right)$.

Again, let V' be the volume of the cone ADC .

Then
$$\begin{aligned} V : V' &= EG^2 : DG^2 \\ &= \frac{BB'^2}{AA'^2} \cdot AG^2 : DG^2. \end{aligned}$$

But $DG^2 : AG \cdot GA' = BB'^2 : AA'^2$.

Therefore
$$\begin{aligned} V : V' &= AG^2 : AG \cdot GA' \\ &= AG : GA'. \end{aligned}$$

$$\begin{aligned}
 \text{It follows that } S &= V' \cdot \frac{AG}{GA'} \left(\frac{3AA'}{2AG} - 1 \right) \\
 &= V' \cdot \frac{\frac{3}{2}AA' - AG}{A'G} \\
 &= V' \cdot \frac{\frac{1}{2}AA' + A'G}{A'G},
 \end{aligned}$$

which is the result stated by Archimedes in Prop. 8.

The result is the same for the segment of a sphere. The proof, of course slightly simpler, is given in Prop. 7.

In the particular case where the segment is half the sphere or spheroid, the relation becomes

$$S = 2V', \quad (\text{Props. 2, 3})$$

and it follows that the volume of the whole sphere or spheroid is $4V'$, where V' is the volume of the cone ABB' ; i.e. the volume of the sphere or spheroid is two-thirds of that of the circumscribing cylinder.

In order now to find the centre of gravity of the segment of a spheroid, we must have the segment acting *where it is*, not at H .

Therefore formula (1) above will not serve. But we found that

$$MN \cdot NQ = (NP^2 + NQ^2),$$

whence $MN^2 : (NP^2 + NQ^2) = (NP^2 + NQ^2) : NQ^2$;

therefore $HA : AN = (NP^2 + NQ^2) : NQ^2$.

(This is separately proved by Archimedes for the sphere in Prop. 9.)

From this we derive, as usual, that the cone AEF and the segment ADC both acting *where they are* balance a volume equal to the cone AEF placed with its centre of gravity at H .

Now the centre of gravity of the cone AEF is on the line AG at a distance $\frac{3}{4}AG$ from A . Let X be the required centre of gravity of the segment. Then, taking moments about A ,

we have $V \cdot HA = S \cdot AX + V \cdot \frac{3}{4}AG$

$$= (4AA' - 3AG) : (6AA' - 4AG)$$

whence

$$AX : XG = (4AA' - 3AG) : (2AA' - AG)$$

$$= (AG + 4A'G) : (AG + 2A'G)$$

which is the result obtained by Archimedes in Prop. 9 for the sphere and in Prop. 10 for the spheroid.

In the case of the hemi-spheroid or hemisphere $AX : XG$ becomes $5 : 3$, a result obtained for the hemisphere in Prop. 6.

The cases of the paraboloid of revolution (Props. 12 and 13) and the hyperboloid of revolution (Prop. 11) follow the same method and it is unnecessary to reproduce them.

For the cases of the two solids dealt with at the end of the treatise the reader must be referred to the propositions themselves. Incidentally, in Prop. 13, Archimedes finds the centre of gravity of the half of a cylinder cut by a plane perpendicular to the axis, or, in other words, the centre of gravity of a circular segment.

We will now take the other treatises in the order in which they appear in the editions.

On the Sphere and Cylinder, I, II.

The main results obtained in Book I are shortly stated in a prefatory letter to Dositheus. Archimedes tells him that they are new, and that he is now publishing them for the first time, in order that mathematicians may be able to examine the proofs and judge of their value. The results are (1) that the surface of a sphere is four times that of a great circle of the sphere, (2) that the surface of any segment of a sphere is equal to a circle the radius of which is equal to the straight line drawn from the vertex of the segment to the circumference of the base, (3) that the volume of a cylinder circumscribing a sphere and with height equal to the diameter of the sphere is $\frac{3}{2}$ of the volume of the sphere, (4) that the surface of the circumscribing cylinder is also $\frac{3}{2}$ of the surface of the sphere. It is of note that, while the first and third of these propositions appear in the book in this order (Props. 33 and 34), the second appears last (Prop. 35).

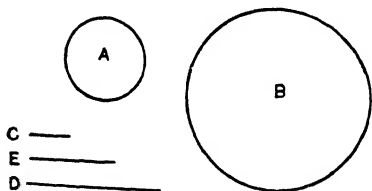
‘from the theorem that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere I conceived the notion that the surface of any sphere is four times as great as a great circle in it; for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius’.

Book I begins with definitions (of ‘concave in the same direction’ as applied to curves or broken lines and surfaces, of a ‘solid sector’ and a ‘solid rhombus’) followed by five Assumptions, all of importance. *Of all lines which have the same extremities the straight line is the least*, and, if there are two curved or bent lines in a plane having the same extremities and concave in the same direction, but one is wholly included by, or partly included by and partly common with, the other, then that which is included is the lesser of the two. Similarly with plane surfaces and surfaces concave in the same direction. Lastly, Assumption 5 is the famous ‘Axiom of Archimedes’, which however was, according to Archimedes himself, used by earlier geometers (Eudoxus in particular), to the effect that *Of unequal magnitudes the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude of the same kind*; the axiom is of course practically equivalent to Eucl. V, Def. 4, and is closely connected with the theorem of Eucl. X. 1.

As, in applying the method of exhaustion, Archimedes uses both circumscribed and inscribed figures with a view to *compressing* them into coalescence with the curvilinear figure to be measured, he has to begin with propositions showing that, given two unequal magnitudes, then, however near the ratio of the greater to the less is to 1, it is possible to find two straight lines such that the greater is to the less in a still less ratio (> 1), and to circumscribe and inscribe similar polygons to a circle or sector such that the perimeter or the area of the circumscribed polygon is to that of the inner in a ratio less than the given ratio (Props. 2-6): also, just as Euclid proves

regular polygon inscribed in a circle, segments with diameters left which are together less than any assigned area, Archimedes has to supplement this (Prop. 6) by proving that, if we increase the number of the sides of a *circumscribed* regular polygon sufficiently, we can make the excess of the area of the polygon over that of the circle less than any given area. Archimedes then addresses himself to the problems of finding the surface of any right cone or cylinder, problems finally solved in Propositions 13 (the cylinder) and 14 (the cone). Circumscribing and inscribing regular polygons to the bases of the cone and cylinder, he erects pyramids and prisms respectively on the polygons as bases and circumscribed or inscribed to the cone and cylinder respectively. In Props. 7 and 8 he finds the surface of the pyramids inscribed and circumscribed to the cone, and in Props. 9 and 10 he proves that the surfaces of the inscribed and circumscribed pyramids respectively (excluding the bases) are less and greater than the surface of the cone (excluding the base). Props. 11 and 12 prove the same thing for prisms inscribed and circumscribed to the cylinder, and in Props. 13 and 14 prove, by the method of exhaustion, that the surface of the cone or cylinder (excluding the bases) is equal to the area of the circle the radius of which is a mean proportional between the 'side' (i.e. generator) of the cone or cylinder and the radius or diameter of the base (i.e. is equal to $\pi r s$ in the case of the cone and $2\pi r s$ in the case of the cylinder, where r is the radius of the base and s a generator). As Archimedes here applies the method of exhaustion for the first time, we will illustrate by the case of the cone (Prop. 14).

Let A be the base of the cone, C a straight line equal



to the radius, D a line equal to a generator of the cone, E a line proportional to C , D , and B a circle with radius equal

For, if S is not equal to B , it must be either greater or less.

I. Suppose $B < S$.

Circumscribe a regular polygon about B , and inscribe a similar polygon in it, such that the former has to the latter a ratio less than $S:B$ (Prop. 5). Describe about A a similar polygon and set up from it a pyramid circumscribing the cone.

$$\begin{aligned}\text{Then} \quad & (\text{polygon about } A) : (\text{polygon about } B) \\ & = C^2 : E^2 \\ & = C : D \\ & = (\text{polygon about } A) : (\text{surface of pyramid}).\end{aligned}$$

Therefore $(\text{surface of pyramid}) = (\text{polygon about } B)$.

But $(\text{polygon about } B) : (\text{polygon in } B) < S : B$;
therefore $(\text{surface of pyramid}) : (\text{polygon in } B) < S : B$.

But this is impossible, since $(\text{surface of pyramid}) > S$, while $(\text{polygon in } B) < B$;
therefore B is not less than S .

II. Suppose $B > S$.

Circumscribe and inscribe similar regular polygons to B such that the former has to the latter a ratio $< B:S$. Inscribe in A a similar polygon, and erect on A the inscribed pyramid.

$$\begin{aligned}\text{Then} \quad & (\text{polygon in } A) : (\text{polygon in } B) = C^2 : E^2 \\ & = C : D \\ & > (\text{polygon in } A) : (\text{surface of pyramid}).\end{aligned}$$

(The latter inference is clear, because the ratio of $C:D$ is greater than the ratio of the perpendiculars from the centre of A and from the vertex of the pyramid respectively on any side of the polygon in A ; in other words, if $\beta < \alpha < \frac{1}{2}\pi$, $\sin \alpha > \sin \beta$.)

Therefore $(\text{surface of pyramid}) > (\text{polygon in } B)$.

But $(\text{polygon about } B) : (\text{polygon in } B) < B : S$,
whence (*a fortiori*)

$$(\text{polygon about } B) : (\text{surface of pyramid}) < B : S,$$

which is impossible, for $(\text{polygon about } B) > B$, while $(\text{surface of pyramid}) < S$.

Therefore B is not greater than S .

Hence S , being neither greater nor less than B , is equal to

Archimedes next addresses himself to the problem of finding the surface and volume of a sphere or a segment thereof, but has to interpolate some propositions about 'solid rhombi' (figures made up of two right cones, unequal or equal, with bases coincident and vertices in opposite directions) the necessity of which will shortly appear.

Taking a great circle of the sphere or a segment of it, he inscribes a regular polygon of an even number of sides bisected

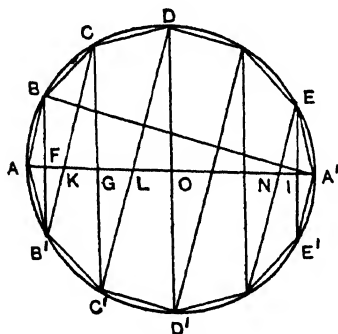


FIG. 1.

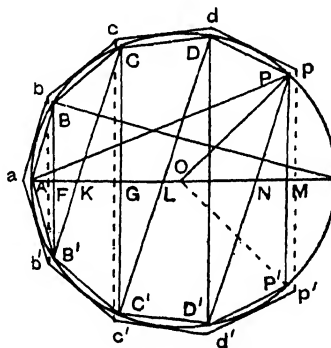


FIG. 2.

by the diameter AA' , and approximates to the surface and volume of the sphere or segment by making the polygon revolve about AA' and measuring the surface and volume of the solid so inscribed (Props. 21-7). He then does the same for a circumscribed solid (Props. 28-32). Construct the inscribed polygons as shown in the above figures. Joining BB' , CC' , CB' , DC' ... we see that BB' , CC' ... are all parallel, and so AB , CB' , DC' ...

Therefore, by similar triangles, $BF:FA = A'B:BA$, and

$$\begin{aligned} BF:FA &= B'F:FK \\ &= CG:GK \end{aligned}$$

(Fig. 1) $(BB' + CC' + \dots + EE') : AA' = A'B : BA$, (Prop. 21)

(Fig. 2) $(BB' + CC' + \dots + \frac{1}{2}PP') : AM = A'B : BA$. (Prop. 22)

When we make the polygon revolve about AA' , the surface of the inscribed figure so obtained is made up of the surfaces of cones and frusta of cones; Prop. 14 has proved that the surface of the cone ABB' is what we should write $\pi \cdot AB \cdot BF$, and Prop. 16 has proved that the surface of the frustum $BCC'B'$ is $\pi \cdot BC (BF + CG)$. It follows that, since $AB = BC = \dots$, the surface of the inscribed solid is

$$\pi \cdot AB \left\{ \frac{1}{2}BB' + \frac{1}{2}(BB' + CC') + \dots \right\},$$

that is, $\pi \cdot AB (BB' + CC' + \dots + EE')$ (Fig. 1), (Prop. 24)

or $\pi \cdot AB (BB' + CC' + \dots + \frac{1}{2}PP')$ (Fig. 2). (Prop. 35)

Hence, from above, the surface of the inscribed solid is $\pi \cdot A'B \cdot AA'$ or $\pi \cdot A'B \cdot AM$, and is therefore less than $\pi \cdot AA'^2$ (Prop. 25) or $\pi \cdot A'A \cdot AM$, that is, $\pi \cdot AP^2$ (Prop. 37).

Similar propositions with regard to surfaces formed by the revolution about AA' of regular circumscribed solids prove that their surfaces are greater than $\pi \cdot AA'^2$ and $\pi \cdot AP^2$ respectively (Props. 28–30 and Props. 39–40). The case of the segment is more complicated because the circumscribed polygon with its sides parallel to $AB, BC \dots DP$ circumscribes the sector POP' . Consequently, if the segment is less than a semicircle, as CAC' , the base of the circumscribed polygon (cc') is on the side of CC' towards A , and therefore the circumscribed polygon leaves over a small strip of the inscribed. This complication is dealt with in Props. 39–40. Having then arrived at circumscribed and inscribed figures with surfaces greater and less than $\pi \cdot AA'^2$ and $\pi \cdot AP^2$ respectively, and having proved (Props. 32, 41) that the surfaces of the circumscribed and inscribed figures are to one another in the duplicate ratio of their sides, Archimedes proceeds to prove formally, by the method of exhaustion, that the surfaces of the sphere and segment are equal to these circles respectively (Props. 33 and 42); $\pi \cdot AA'^2$ is of course equal to four times the great circle of the sphere. The segment is, for convenience, taken to be

less than a hemisphere, and Prop. 43 proves that the formula applies also to a segment greater than a hemisphere.

As regards the volumes different considerations involving 'solid rhombi' come in. For convenience Archimedes in the case of the whole sphere, an inscribed polygon of $4n$ sides (Fig. 1). It is easily seen that the solid figure formed by its revolution is made up of the following: first, the solid rhombus formed by the revolution of the quadrilateral AOB (the volume of this is shown to be equal to the cone with base equal to the surface of the cone ABB' and height equal to the perpendicular from O on AB , Prop. 18); secondly, the extinguisher-shaped figure formed by the revolution of the triangle BOC about AA' (this figure is equal to the difference between two solid rhombi formed by the revolution of TCO and $TCOC'$ respectively about AA' , where T is the point of intersection of CB , $C'B'$ produced with $A'A$ produced); this difference is proved to be equal to a cone with base equal to the surface of the frustum of a cone described by BC in its revolution and height equal to p the perpendicular from O on BC , Prop. 20); and so on; finally, the figure formed by the revolution of the triangle COD about AA' is the difference between a cone and a solid rhombus, which is proved equal to a cone with base equal to the surface of the frustum of a cone described by CD in its revolution and height p (Prop. 20). Consequently, by addition, the volume of the whole solid of revolution is equal to the cone with base equal to its surface and height p (Prop. 26). But the whole of the surface of the solid is less than $4\pi r^2$, and $p < r$; therefore the volume of the inscribed solid is less than four times the cone with base πr^2 and height r (Prop. 27).

It is then proved in a similar way that the volume of the solid of revolution of the similar circumscribed polygon of $4n$ sides gives a solid the volume of which is *greater* than four times the same cone (Props. 28–31 Cor.). Lastly, the volumes of the circumscribed and inscribed figures are to one another in the triplicate ratio of their surfaces.

same chain of argument (Props. 38, 40 Corr., 41 and 42), proves (Prop. 44) that the volume of the *sector* of the sphere bounded by the surface of the segment is equal to a cone with base equal to the surface of the segment and height equal to the radius, i. e. the cone with base $\pi \cdot AP^2$ and height r (Fig. 2).

It is noteworthy that the proportions obtained in Props. 21, 22 (see p. 39 above) can be expressed in trigonometrical form. If $4n$ is the number of the sides of the polygon inscribed in the circle, and $2n$ the number of the sides of the polygon inscribed in the segment, and if the angle AOP is denoted by α , the trigonometrical equivalents of the proportions are respectively

$$(1) \quad \sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \dots + \sin (2n-1) \frac{\pi}{2n} = \cot \frac{\pi}{4n};$$

$$(2) \quad 2 \left\{ \sin \frac{\alpha}{n} + \sin \frac{2\alpha}{n} + \dots + \sin (n-1) \frac{\alpha}{n} \right\} + \sin \alpha \\ = (1 - \cos \alpha) \cot \frac{\alpha}{2n}.$$

Thus the two proportions give in effect a summation of the series

$$\sin \theta + \sin 2\theta + \dots + \sin (n-1)\theta,$$

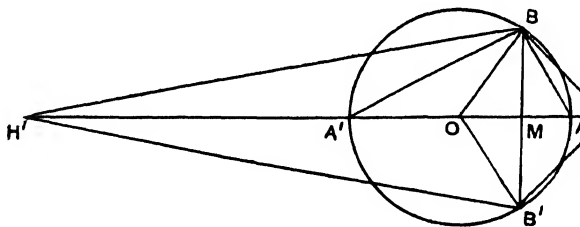
both generally where $n\theta$ is equal to any angle α less than π and in the particular case where n is even and $\theta = \pi/n$. Props. 24 and 35 prove that the areas of the circles equal to the surfaces of the solids of revolution described by the polygons inscribed in the sphere and segment are the above series multiplied by $4\pi r^2 \sin \frac{\pi}{4n}$ and $\pi r^2 \cdot 2 \sin \frac{\alpha}{2n}$ respectively and are therefore $4\pi r^2 \cos \frac{\pi}{4n}$ and $\pi r^2 \cdot 2 \cos \frac{\alpha}{2n} (1 - \cos \alpha)$ respectively. Archimedes's results for the surfaces of the sphere and segment, $4\pi r^2$ and $2\pi r^2(1 - \cos \alpha)$, are the limiting values of these expressions when n is indefinitely increased and when therefore $\cos \frac{\pi}{4n}$ and $\cos \frac{\alpha}{2n}$ become unity. And the two series multiplied by $4\pi r^2 \sin \frac{\pi}{4n}$ and

$\pi r^2 \cdot 2 \sin \frac{\alpha}{2n}$ respectively are (when n is indefinitely increased) precisely what we should represent by the integrals

$$4\pi r^2 \cdot \frac{1}{2} \int_0^\pi \sin \theta d\theta, \text{ or } 4\pi r^2,$$

and $\pi r^2 \cdot \int_0^\alpha 2 \sin \theta d\theta, \text{ or } 2\pi r^2(1 - \cos \alpha).$

Book II contains six problems and three theorems. Theorem Prop. 2 completes the investigation of the volume of any segment of a sphere, Prop. 44 of Book I having brought us to the volume of the corresponding spherical cap. Let ABB' be a segment of a sphere cut off by a plane perpendicular to AA' , we learnt in I. 44 that the volume of the



$OBAB'$ is equal to the cone with base equal to the area of the segment and height equal to the radius, i.e. $\frac{1}{3}\pi r^2$, where r is the radius. The volume of the segment is therefore

$$\frac{1}{3}\pi \cdot AB^2 \cdot r - \frac{1}{3}\pi \cdot BM^2 \cdot OM.$$

Archimedes wishes to express this as a cone with the same base as that of the segment. Let AM , the height of the segment, $= h$.

Now $AB^2 : BM^2 = A'A : A'M = 2r : (2r - h).$

Therefore

$$\frac{1}{3}\pi (AB^2 \cdot r - BM^2 \cdot OM) = \frac{1}{3}\pi \cdot BM^2 \left\{ \frac{2r^2}{2r - h} - (r - h) \right\}$$

of the required cone,

$$HM:AM = (OA' + A'M):A'M, \quad (1)$$

and similarly the cone equal to the segment $A'BB'$ has the height $H'M$, where

$$H'M:A'M = (OA + AM):AM. \quad (2)$$

His proof is, of course, not in the above form but purely geometrical.

This proposition leads to the most important proposition in the Book, Prop. 4, which solves the problem *To cut a given sphere by a plane in such a way that the volumes of the segments are to one another in a given ratio.*

Cubic equation arising out of II. 4.

If $m:n$ be the given ratio of the cones which are equal to the segments and the heights of which are h, h' , we have

$$h\left(\frac{3r-h}{2r-h}\right) = \frac{m}{n} h' \left(\frac{3r-h'}{2r-h'}\right),$$

and, if we eliminate h' by means of the relation $h+h' = 2r$, we easily obtain the following cubic equation in h ,

$$h^3 - 3h^2r + \frac{4m}{m+n} r^3 = 0.$$

Archimedes in effect reduces the problem to this equation, which, however, he treats as a particular case of the more general problem corresponding to the equation

$$(r+h):b = c^2:(2r-h)^2,$$

where b is a given length and c^2 any given area,

or $x^2(a-x) = bc^2$, where $x = 2r-h$ and $3r = a$.

Archimedes obtains his cubic equation with one unknown by means of a *geometrical* elimination of H, H' from the equation $HM = \frac{m}{n} \cdot H'M$, where $HM, H'M$ have the values determined by the proportions (1) and (2) above, after which the one variable point M remaining corresponds to the one unknown of the cubic equation. His method is, first, to find

values for each of the ratios $A'H':H'M$ and $H'H:A'H'$ which are alike independent of H , H' and then, secondly, to equate the ratio compounded of these two to the known value of the ratio $HH':H'M$.

(α) We have, from (2),

$$A'H':H'M = OA:(OA + AM).$$

(β) From (1) and (2), *separando*,

$$AH:AM = OA':A'M,$$

$$A'H':A'M = OA:AM.$$

Equating the values of the ratio $A'M:AM$ given by (4),

$$\begin{aligned} \text{we have} \quad OA':AH &= A'H':OA \\ &= OH':OH, \end{aligned}$$

$$\text{whence} \quad HH':OH' = OH':A'H', \quad (\text{since } OA = OH')$$

$$\text{or} \quad HH' \cdot A'H' = OH'^2,$$

$$\text{so that} \quad HH':A'H' = OH'^2:A'H'^2.$$

$$\text{But, by (5),} \quad OA':A'H' = AM:A'M,$$

$$\text{and, componendo,} \quad OH':A'H' = AA':A'M.$$

By substitution in (6),

$$HH':A'H' = AA'^2:A'M^2.$$

Compounding with (3), we obtain

$$HH':H'M = (AA'^2:A'M^2) \cdot (OA:OA + AM).$$

[The algebraical equivalent of this is

$$\frac{m+n}{n} = \frac{4r^3}{(2r-h)^2(r+h)},$$

$$\text{which reduces to} \quad \frac{m+n}{m} = \frac{4r^3}{3h^2r-h^3},$$

$$\text{or} \quad \frac{13}{21} = \frac{4m}{3h^2r-h^3} \quad \frac{13}{21} = \frac{4m}{3h^2r-h^3}$$

E so that $AD:DE = HH':H'M$ or $(m+n):n$. We have then $OA = AD$ and $OA + AM = MD$, so that (8) reduces to

$$AD:DE = (AA'^2:A'M^2) \cdot (AD:MD),$$

or
$$MD:DE = AA'^2:A'M^2.$$

Now, says Archimedes, D is given, since $AD = OA$. Also, $AD:DE$ being a given ratio, DE is given. Hence the problem reduces itself to that of dividing $A'D$ into two parts at M such that

$$MD:(\text{a given length}) = (\text{a given area}):A'M^2.$$

That is, the generalized equation is of the form

$$x^2(a-x) = bc^2, \text{ as above.}$$

(i) Archimedes's own solution of the cubic.

Archimedes adds that, 'if the problem is propounded in this general form, it requires a $\delta\iota\omicron\rho\iota\sigma\mu\acute{o}\varsigma$ [i.e. it is necessary to investigate the limits of possibility], but if the conditions are added which exist in the present case [i.e. in the actual problem of Prop. 4], it does not require a $\delta\iota\omicron\rho\iota\sigma\mu\acute{o}\varsigma$ ' (in other words, a solution is always possible). He then promises to give 'at the end' an analysis and synthesis of both problems [i.e. the $\delta\iota\omicron\rho\iota\sigma\mu\acute{o}\varsigma$ and the problem itself]. The promised solutions do not appear in the treatise as we have it, but Eutocius gives solutions taken from 'an old book' which he managed to discover after laborious search, and which, since it was partly written in Archimedes's favourite Doric, he with fair reason assumed to contain the missing *addendum* by Archimedes.

In the Archimedean fragment preserved by Eutocius the above equation, $x^2(a-x) = bc^2$, is solved by means of the intersection of a parabola and a rectangular hyperbola, the equations of which may be written thus

$$x^2 = \frac{c^2}{a}y, \quad (a-x)y = ab.$$

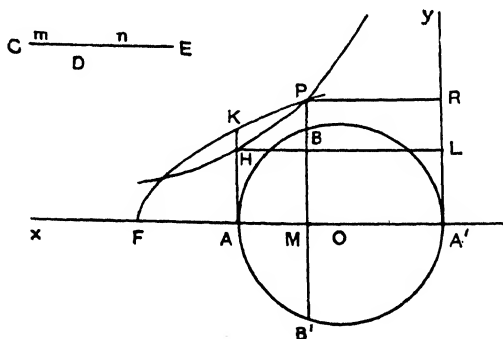
other hand $bc^2 < \frac{m}{27}a^3$, it is proved that there are solutions. In the particular case arising in Prop. 4 it is assumed that the condition for a real solution is satisfied, and the expression corresponding to bc^2 is $\frac{m}{m+n} 4r^3$, and it is necessary that $\frac{m}{m+n} 4r^3$ should be not greater than $4r^3$, which is obviously the case.

(ii) Solution of the cubic by Dionysodorus.

It is convenient to add here that Eutocius gives, in connection with the solution by Archimedes, two other solutions of the problem. One, by Dionysodorus, solves the cubic equation in the less general form in which it is required for Archimedes' proposition. This form, obtained from (8) above, by putting $A'M = x$, is

$$4r^2 : x^2 = (3r - x) : \frac{n}{m+n} r,$$

and the solution is obtained by drawing the parabola



the rectangular hyperbola which we should represent by the equations

$$\frac{n}{m+n} r (3r - x) = y^2 \quad \text{and} \quad \frac{n}{m+n} 2r^2 = xy,$$

referred to $A'A$ and the perpendicular to it through O , of x, y respectively.

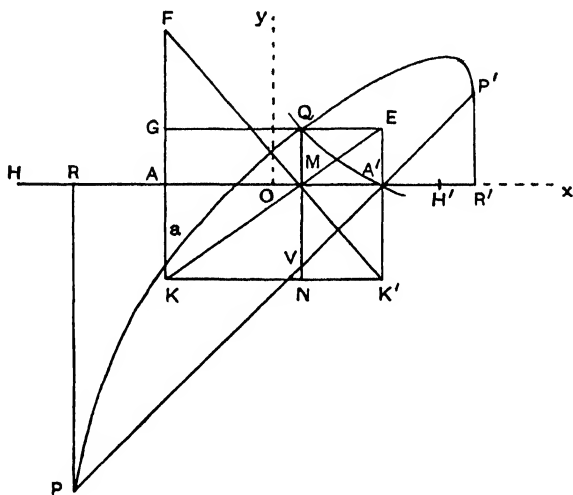
(We make FA equal to OA , and draw the perpendicular AH of such a length that

$$FA : AH = CE : ED = (m+n) : n.)$$

Diocles proceeded in a different manner, satisfying, by a geometrical construction, not the derivative cubic equation, but the three simultaneous relations which hold in Archimedes's proposition, namely

$$\left. \begin{aligned} HM : H'M &= m : n \\ HA : h &= r : h' \\ H'A' : h' &= r : h \end{aligned} \right\},$$

with the slight generalization that he substitutes for r in these equations another length α .



The problem is, given a straight line AA' , a ratio $m:n$, and another straight line $AK (= \alpha)$, to divide AA' at a point M and at the same time to find two points H, H' on AA' produced such that the above relations (with α in place of r) hold.

The analysis leading to the construction is very ingenious. Place $AK (= \alpha)$ at right angles to AA' , and draw $A'K'$ equal and parallel to it.

Suppose the problem solved, and the points M, H, H' all found.

Join KM , produce it, and complete the rectangle $KGEK'$.

Draw QMN through M parallel to AK . Produce AK to meet KG produced in F .

By similar triangles,

$$FA : AM = K'A' : A'M, \text{ or } FA : h = a : h$$

whence $FA = AH$ (k , suppose).

Similarly $A'E = A'H'$ (k' , suppose).

Again, by similar triangles,

$$\begin{aligned} (FA + AM) : (A'K' + A'M) &= AM : A'M \\ &= (AK + AM) : (EA' + AM) \end{aligned}$$

$$\text{or} \quad (k + h) : (a + h') = (a + h) : (k' + h'),$$

$$\text{i. e.} \quad (k + h) (k' + h') = (a + h) (a + h').$$

Now, by hypothesis,

$$\begin{aligned} m : n &= (k + h) : (k' + h') \\ &= (k + h) (k' + h') : (k' + h')^2 \\ &= (a + h) (a + h') : (k' + h')^2 \text{ [by (1)]}. \end{aligned}$$

Measure AR , $A'R'$ on AA' produced both ways equal to AM . Draw RP , $R'P'$ at right angles to RR' as shown in the diagram. Measure along MN the length MV equal to MA' or AR and draw PI' through V , A' to meet RP , $R'P'$.

$$\text{Then} \quad QV = k' + h', \quad P'V = \sqrt{2} (a + h'),$$

$$PV = \sqrt{2} (a + h),$$

$$\text{whence} \quad PV \cdot P'V = 2 (a + h) (a + h');$$

and, from (2) above,

$$\begin{aligned} 2m : n &= 2 (a + h) (a + h') : (k' + h')^2 \\ &= PV \cdot P'V : QV^2. \end{aligned}$$

Therefore Q is on an ellipse in which PI' is a diameter and QV is an ordinate to it.

How this ingenious analysis was suggested it is not possible to say. It is the equivalent of reducing the four unknowns h, h', k, k' to two, by putting $h = r + x$, $h' = r - x$ and $k' = y$, and then reducing the given relations to two equations in x, y , which are coordinates of a point in relation to Ox, Oy as axes, where O is the middle point of AA' , and Ox lies along OA' , while Oy is perpendicular to it.

Our original relations (p. 47) give

$$y = k' = \frac{ah'}{h} = a \frac{r-x}{r+x}, \quad k = \frac{ah}{h'} = a \frac{r+x}{r-x}, \quad \text{and} \quad \frac{m}{n} = \frac{h+k}{h'+k'}.$$

We have at once, from the first two equations,

$$ky = a \frac{r+x}{r-x} y = a^2,$$

whence $(r+x)y = a(r-x),$

and $(x+r)(y+a) = 2ra,$

which is the rectangular hyperbola (4) above.

Again,
$$\frac{m}{n} = \frac{h+k}{h'+k'} = \frac{(r+x) \left(1 + \frac{a}{r-x}\right)}{(r-x) \left(1 + \frac{a}{r+x}\right)},$$

whence we obtain a cubic equation in x ,

$$(r+x)^2(r+a-x) = \frac{m}{n}(r-x)^2(r+a+x),$$

which gives

$$\frac{m}{n}(r-x)^2 \left(\frac{r+a+x}{r+x} \right)^2 = (r+a)^2 - x^2.$$

But $\frac{y}{r-x} = \frac{a}{r+x}$, whence $\frac{y+r-x}{r-x} = \frac{r+a+x}{r+x},$

and the equation becomes

To return to Archimedes. Book II of our treatise contains the following further problems: To find a sphere equal to a given cylinder (Prop. 1), solved by reduction to the finding of two mean proportionals; to cut a sphere by a plane so that the segments having their surfaces in a given ratio shall be found, which is easy (by means of I. 42, 43); given two spheres, to find a third segment of a sphere similar to one of the given segments and having its surface equal to the other (Prop. 6); the same problem with volume instead of surface (Prop. 5), which is again reduced to the finding of two mean proportionals; from a given sphere to cut a segment having a given ratio to the cone with the same base and equal height (Prop. 7). The Book contains also two interesting theorems. If a sphere be cut by a plane into two segments, the greater of which has its surface equal to S' , and its volume equal to V , while S' , V' are the surface and volume of the lesser, then $V:V' < S^2:S'^2$ (Prop. 8): and, of all segments of spheres which have equal surfaces, the hemisphere is the greatest (Prop. 9).

Measurement of a Circle.

The book on the *Measurement of a Circle* contains only propositions only, and is not in its original form (as the treatise *On the Sphere and Cylinder* is) without any trace of the Doric dialect in which it was written: it may be only a fragment of a larger work. The three propositions which survive prove (1) that the area of a circle is equal to that of a right-angled triangle whose base is the circumference, and whose height is the perpendicular from the centre to the circumference, (2) that the area of a circle is to the square on its diameter as 11 to 14 (the position is, however, unsatisfactory, and it can be replaced by Archimedes before Prop. 3, on which he shows that the ratio of the circumference of a circle to its diameter is between $\frac{22}{7}$ and $\frac{223}{71}$).

sides continually doubled, beginning from a square, (b) by circumscribing a similar set of regular polygons beginning from a square, it being shown that, if the number of the sides of these polygons be continually doubled, more than half of the portion of the polygon outside the circle will be taken away each time, so that we shall ultimately arrive at a circumscribed polygon greater than the circle by a space less than any assigned area.

Prop. 3, containing the arithmetical approximation to π , is the most interesting. The method amounts to calculating approximately the perimeter of two regular polygons of 96 sides, one of which is circumscribed, and the other inscribed, to the circle; and the calculation starts from a greater and a lesser limit to the value of $\sqrt{3}$, which Archimedes assumes without remark as known, namely

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}.$$

How did Archimedes arrive at these particular approximations? No puzzle has exercised more fascination upon writers interested in the history of mathematics. De Lagny, Mollweide, Buzengeiger, Hauber, Zeuthen, P. Tannery, Heilermann, Hultsch, Hunrath, Wertheim, Bobynin: these are the names of some of the authors of different conjectures. The simplest supposition is certainly that of Hunrath and Hultsch, who suggested that the formula used was

$$a \pm \frac{b}{2a} > \sqrt{a^2 \pm b} > a \pm \frac{b}{2a \pm 1},$$

where a^2 is the nearest square number above or below $a^2 \pm b$, as the case may be. The use of the first part of this formula by Heron, who made a number of such approximations, is proved by a passage in his *Metrica*¹, where a rule equivalent to this is applied to $\sqrt{720}$; the second part of the formula is used by the Arabian Alkarkhī (eleventh century) who drew from Greek sources, and one approximation in Heron may be

and Zeuthen) is that the successive solutions in the equations

$$\left. \begin{aligned} x^2 - 3y^2 &= 1 \\ x^2 - 3y^2 &= -2 \end{aligned} \right\}$$

may have been found in a similar way to the equations $x^2 - 2y^2 = \pm 1$ given by Theon of Smyrna and the Pythagoreans. The rest of the suggestions are due in the most part to the use of the method of continued fractions, more or less disguised.

Applying the above formula, we easily find

$$2 - \frac{1}{4} > \sqrt{3} > 2 - \frac{1}{3},$$

or

$$\frac{7}{4} > \sqrt{3} > \frac{5}{3}.$$

Next, clearing of fractions, we consider 5 as a denominator in the fraction $\frac{1}{3}$, and we have

$$5 + \frac{2}{15} > 3\sqrt{3} > 5 + \frac{1}{15},$$

whence

$$\frac{76}{15} > \sqrt{3} > \frac{77}{15}.$$

Clearing of fractions again, and taking 26 as a denominator in the fraction $\frac{1}{15}$, we have

$$26 - \frac{1}{52} > 15\sqrt{3} > 26 - \frac{1}{51},$$

which reduces to

$$\frac{1351}{780} > \sqrt{3} > \frac{1352}{780}.$$

Archimedes first takes the case of the circumscribed polygon. Let CA be the tangent at A to a circular arc with center O . Make the angle AOC equal to one-third of a right angle. Bisect the angle AOC by OD , the angle AOD by OE , the angle AOE by OF , and the angle AOF by OG . Draw AG to AH , making AH equal to AG . The angle G is equal to the angle FOA which is $\frac{1}{24}$ th of a right angle.

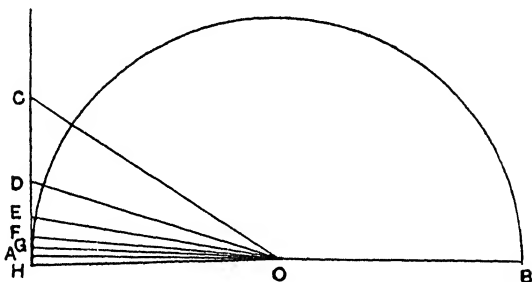
And, since OD bisects the angle COA ,

$$CO : OA = CD : DA,$$

so that $(CO + OA) : OA = CA : DA$,

or $(CO + OA) : CA = OA : AD$.

Hence . $OA : AD > 571 : 153$, by (1) and (2).



$$\begin{aligned} \text{And } OD^2 : AD^2 &= (OA^2 + AD^2) : AD^2 \\ &> (571^2 + 153^2) : 153^2 \\ &> 349450 : 23409. \end{aligned}$$

Therefore, says Archimedes,

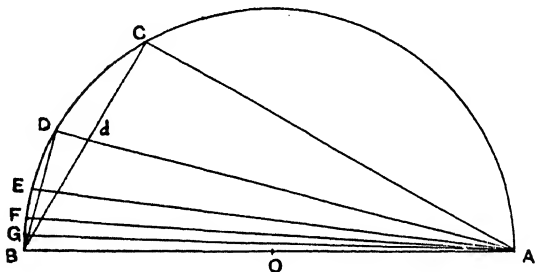
$$OD : DA > 591\frac{1}{8} : 153.$$

Next, just as we have found the limit of $OD : AD$ from $OC : CA$ and the limit of $OA : AC$, we find the limits of $OA : AE$ and $OE : AE$ from the limits of $OD : DA$ and $OA : AD$, and so on. This gives ultimately the limit of $OA : AG$.

Dealing with the inscribed polygon, Archimedes gets a similar series of approximations. ABC being a semicircle, the angle BAC is made equal to one-third of a right angle. Then,

Now the triangles ADB , BDd , ACd are similar;
 therefore $AD : DB = BD : Dd = AC : Cd$
 $= AB : Bd$, since AD bisects $\angle BAC$
 $= (AB + AC) : (Bd + Cd)$
 $= (AB + AC) : BC$.

But $AC : CB < 1351 : 780$,
 while $BA : BC = 2 : 1 = 1560 : 780$.
 Therefore $AD : DB < 2911 : 780$.



Hence $AB^2 : BD^2 < (2911^2 + 780^2) : 780^2$
 $< 9082321 : 608400$,
 and, says Archimedes,

$$AB : BD < 3013\frac{3}{4} : 780.$$

Next, just as a limit is found for $AD : DB$ and $AB : BD$ from $AB : BC$ and the limit of $AC : CB$, so we find limits $AE : EB$ and $AB : BE$ from the limits of $AB : BD$ and $AD : DB$ and so on, and finally we obtain the limit of $AB : BG$.

We have therefore in both cases two series of terms $a_0, a_1, a_2, \dots, a_n$ and $b_0, b_1, b_2, \dots, b_n$, for which the rule of formation is

$$a_1 = a_0 + b_0, a_2 = a_1 + b_1, \dots,$$

where $b_1 = \sqrt{(a_1^2 + c^2)}$, $b_2 = \sqrt{(a_2^2 + c^2)}$... ;

and in the first case

The series of values found by Archimedes are shown in the following table:

b	c	n	a	b	c
.306	153	0	1351	1560	780
$[\sqrt{(571^2 + 153^2)}]$	153	1	2911	$< \sqrt{(2911^2 + 780^2)}$	780
$> 591\frac{1}{8}$				$< 3013\frac{1}{2}\frac{1}{4}$	
$\sqrt{\{(1162\frac{1}{8})^2 + 153^2\}}$	153	2	$5924\frac{1}{2}\frac{1}{4}$...	780)*
$> 1172\frac{1}{8}$			$\left\{ \begin{array}{l} 1823 \\ \left\{ \begin{array}{l} < \sqrt{(1823^2 + 240^2)} \\ < 1838\frac{9}{11} \end{array} \right. \end{array} \right.$	$\left\{ \begin{array}{l} 240 \\ 240 \end{array} \right.$	
$\sqrt{\{(2334\frac{1}{4})^2 + 153^2\}}$	153	3	$3661\frac{9}{11}$...	240†
$> 2339\frac{1}{4}$			$\left\{ \begin{array}{l} 1007 \\ \left\{ \begin{array}{l} < \sqrt{(1007^2 + 66^2)} \\ < 1009\frac{1}{6} \end{array} \right. \end{array} \right.$	$\left\{ \begin{array}{l} 66 \\ 66 \end{array} \right.$	
	153	4	$2016\frac{1}{6}$	$\left\{ \begin{array}{l} < \sqrt{\{(2016\frac{1}{6})^2 + 66^2\}} \\ < 2017\frac{1}{4} \end{array} \right.$	66

bearing in mind that in the first case the final ratio is the ratio $OA : AG = 2 OA : GH$, and in the second case final ratio $b_4 : c$ is the ratio $AB : BG$, while GH in the first case and BG in the second are the sides of regular polygons of 6 sides circumscribed and inscribed respectively, we have only

$$\frac{96 \times 153}{4673\frac{1}{2}} > \pi > \frac{96 \times 66}{2017\frac{1}{4}}.$$

Archimedes simply infers from this that

$$3\frac{1}{7} > \pi > 3\frac{1}{7}\frac{9}{11}.$$

As a matter of fact $\frac{96 \times 153}{4673\frac{1}{2}} = 3\frac{667\frac{1}{2}}{4673\frac{1}{2}}$, and $\frac{667\frac{1}{2}}{4672\frac{1}{2}} = \frac{1}{7}$.

It should also be observed that $3\frac{1}{7}\frac{9}{11} = 3 + \frac{1}{7 + \frac{1}{10}}$, and it may have been arrived at by a method equivalent to developing the fraction $\frac{6336}{2017\frac{1}{4}}$ in the form of a continued fraction.

It should be noted that in the text as we have it the values

and then that $OD:DA > 591\frac{1}{8}:153$. At the points * and † in the table Archimedes simplifies the ratio $a_3:c$ before calculating b_2, b_3 respectively, by multiplying term in the first case by $\frac{4}{13}$ and in the second case by $\frac{1}{13}$. He gives no explanation of the exact figure taken as an approximation to the square root in each case, or the method by which he obtained it. We may, however, conjecture that the method amounted to the use of the formula $x^2 = a^2 \pm 2ab + b^2$, much as our method of extracting the square root also depends upon it.

We have already seen (vol. i, p. 232) that, according to Heron, Archimedes made a still closer approximation to the value of π .

On Conoids and Spheroids.

The main problems attacked in this treatise are, in Archimedes's manner, stated in his preface addressed to Dositheos, which also sets out the premisses with regard to the figures in question. These premisses consist of definitions and obvious inferences from them. The figures are (1) the *angled conoid* (paraboloid of revolution), (2) the *obtuse-angled conoid* (hyperboloid of revolution), and (3) the *spheroid* (a) the *oblong*, described by the revolution of an ellipse about its 'greater diameter' (major axis), (b) the *flat*, described by the revolution of an ellipse about its 'lesser diameter' (minor axis). Other definitions are those of the *vertex* and *axis* of a figure or segments thereof, the vertex of a segment being the point of contact of the tangent plane to the surface at that point is parallel to the base of the segment. The *centre* is recognized in the case of the spheroid; what corresponds to the centre in the case of the hyperboloid is the 'vertex of the enveloping cone' (described by the revolution of the hyperbola about its asymptotes). Archimedes calls the 'nearest lines to the section of the obtuse-angled cone', i.e. the asymptotes of the hyperbola, and the line between this point and the vertex of the hyperboloid or segment is called, not the axis or diameter, but the line 'adjacent to the axis'. The axis of the segment is the case of the paraboloid the line through the vertex and the vertex of the segment parallel to the axis of the paraboloid, in

the segment and produced, and in the case of the spheroids the line joining the points of contact of the two tangent planes parallel to the base of the segment. Definitions are added of a 'segment of a cone' (the figure cut off towards the vertex by an elliptical, not circular, section of the cone) and a 'frustum of a cylinder' (cut off by two parallel elliptical sections).

Props. 1 to 18 with a Lemma at the beginning are preliminary to the main subject of the treatise. The Lemma and Props. 1, 2 are general propositions needed afterwards. They include propositions in summation,

$$2\{a + 2a + 3a + \dots + na\} > n \cdot na > 2\{a + 2a + \dots + (n-1)a\} \quad (\text{Lemma})$$

(this is clear from $S_n = \frac{1}{2}n(n+1)a$);

$$\begin{aligned} (n+1)(na)^2 + a(a + 2a + 3a + \dots + na) \\ = 3\{a^2 + (2a)^2 + (3a)^2 + \dots + (na)^2\}; \end{aligned} \quad (\text{Lemma to Prop. 2})$$

whence (Cor.)

$$\begin{aligned} 3\{a^2 + (2a)^2 + (3a)^2 + \dots + (na)^2\} > n(na)^2 \\ > 3\{a^2 + (2a)^2 + \dots + (\overline{n-1}a)^2\}; \end{aligned}$$

lastly, Prop. 2 gives limits for the sum of n terms of the series $ax + x^2$, $a \cdot 2x + (2x)^2$, $a \cdot 3x + (3x)^2$, ..., in the form of inequalities of ratios, thus:

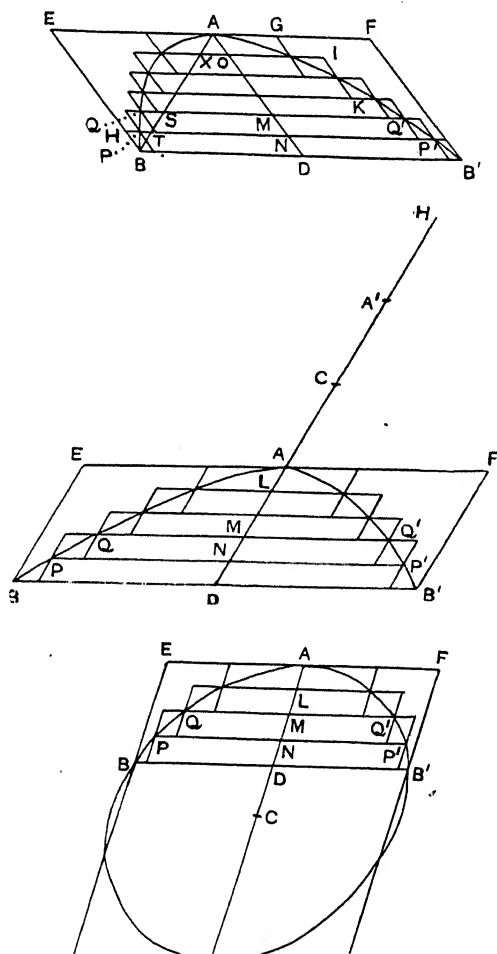
$$\begin{aligned} n\{a \cdot nx + (nx)^2\} : \Sigma_1^{n-1}\{a \cdot rx + (rx)^2\} \\ > (a + nx) : (\tfrac{1}{2}a + \tfrac{1}{2}nx) \\ > n\{a \cdot nx + (nx)^2\} : \Sigma_1^n\{a \cdot rx + (rx)^2\}. \end{aligned}$$

Prop. 3 proves that, if QQ' be a chord of a parabola bisected at V by the diameter PV , then, if PV be of constant length, the areas of the triangle PQQ' and of the segment PQQ' are also constant, whatever be the direction of QQ' ; to prove it Archimedes assumes a proposition 'proved in the conics' and by no means easy, namely that, if QD be perpendicular to PV , and if p, p_a be the parameters corresponding to the ordinates parallel to QQ' and the principal ordinates respectively, then

$$QV^2 : QD^2 = p : p_a.$$

Props. 4-6 deal with the area of an ellipse, which is, in the

the section which is the base of the segment, and which is a circle or an ellipse according as the said base is or is not at right angles to the axis; the plane of the paper cuts the segment in a diameter of the circle or an axis of the ellipse as may be.



segment is an ellipse in which BB' is an axis, and its plane is at right angles to the plane of the paper, which passes through the axis of the solid and cuts it in a parabola, a hyperbola, or an ellipse respectively. The axis of the segment is cut into a number of equal parts in each case, and planes are drawn through each point of section parallel to the base, cutting the solid in ellipses, similar to the base, in which PP' , QQ' , &c. are the axes. Describing frusta of cylinders with axis AD and passing through these elliptical sections respectively, we draw the circumscribed and inscribed solids consisting of these frusta. It is evident that, beginning from A , the first inscribed frustum is equal to the first circumscribed frustum, the second to the second, and so on, but there is one more circumscribed frustum than inscribed, and the difference between the circumscribed and inscribed solids is equal to the *last frustum* of which the base is the base, and ND is the axis. Since ND can be made as small as we please, the difference between the circumscribed and inscribed solids can be made less than any assigned number whatever. Hence we have the requirements for applying the method of exhaustion.

Consider now separately the cases of the paraboloid, hyperboloid and the spheroid.

I. The *paraboloid* (Props. 20–22).

The frustum the base of which is the ellipse in which the axis is proportional to PP'^2 or PN^2 , i.e. proportional to AN . Suppose that the axis AD ($=c$) is divided into n equal parts. Archimedes compares each frustum in the inscribed and circumscribed figure with the frustum of the whole cylinder BF cut off by the same planes. Thus

$$\begin{aligned} (\text{first frustum in } BF) : (\text{first frustum in inscribed figure}) &= BD^2 : PN^2 \\ &= AD : AN \\ &= BD : TN. \end{aligned}$$

Similarly

$$\begin{aligned} (\text{second frustum in } BF) : (\text{second in inscribed figure}) &= HN : SM, \end{aligned}$$

and so on. The last frustum in the cylinder BF has no

correspond to it in the inscribed figure, and we should write the ratio as $(BD : \text{zero})$.

Archimedes concludes, by means of a lemma in proportions forming Prop. 1, that

$$\begin{aligned} (\text{frustum } BF) : (\text{inscribed figure}) \\ &= (BD + HN + \dots) : (TN + SM + \dots + XO) \\ &= n^2k : (k + 2k + 3k + \dots + \overline{n-1}k), \end{aligned}$$

where $XO = k$, so that $BD = nk$.

In like manner, he concludes that

$$\begin{aligned} (\text{frustum } BF) : (\text{circumscribed figure}) \\ &= n^2k : (k + 2k + 3k + \dots + nk). \end{aligned}$$

But, by the Lemma preceding Prop. 1,

$$k + 2k + 3k + \dots + \overline{n-1}k < \frac{1}{2}n^2k < k + 2k + 3k + \dots + nk,$$

whence

$$(\text{frustum } BF) : (\text{inscr. fig.}) > 2 > (\text{frustum } BF) : (\text{circumscr. fig.}).$$

This indicates the desired result, which is then confirmed by the method of exhaustion, namely that

$$(\text{frustum } BF) = 2 (\text{segment of paraboloid}),$$

or, if V be the volume of the 'segment of a cone', with vertex A and base the same as that of the segment,

$$(\text{volume of segment}) = \frac{3}{2}V.$$

Archimedes, it will be seen, proves in effect that, if k be indefinitely diminished, and n indefinitely increased, while nk remains equal to c , then

$$\text{limit of } k \{k + 2k + 3k + \dots + (n-1)k\} = \frac{1}{2}c^2,$$

that is, in our notation,

$$\int_0^c x dx = \frac{1}{2}c^2.$$

Prop. 23 proves that the volume is constant for a given length of axis AD , whether the segment is cut off by a plane perpendicular or not perpendicular to the axis, and Prop. 24 shows that the volumes of two segments are as the squares on their axes.

II. In the case of the *hyperboloid* (Props. 25, 26) let the AD be divided into n parts, each of length h , and let AA' be divided into n parts, each of length b . Then the ratio of the volume of the frustum of a cylinder to the volume of the frustum of the ellipse of which any double ordinate QQ' is an axis is equal to the ratio of the volume of the corresponding portion of the whole frustum of the cylinder to the volume of the whole frustum. This ratio takes a different form; for, if $AM = rh$, we have

$$\begin{aligned} & (\text{frustum in } BF) : (\text{frustum on base } QQ') \\ &= BD^2 : QM^2 \\ &= AD \cdot A'D : AM \cdot A'M \\ &= \{a \cdot nh + (nh)^2\} : \{a \cdot rh + (rh)^2\}. \end{aligned}$$

By means of this relation Archimedes proves that

$$(\text{frustum } BF) : (\text{inscribed figure})$$

$$= n \{a \cdot nh + (nh)^2\} : \Sigma_1^{n-1} \{a \cdot rh + (rh)^2\}$$

and

$$(\text{frustum } BF) : (\text{circumscribed figure})$$

$$= n \{a \cdot nh + (nh)^2\} : \Sigma_1^n \{a \cdot rh + (rh)^2\}$$

But, by Prop. 2,

$$\begin{aligned} n \{a \cdot nh + (nh)^2\} : \Sigma_1^{n-1} \{a \cdot rh + (rh)^2\} &> (a + nh) : (\tfrac{1}{2}a + \tfrac{1}{2}b) \\ &> n \{a \cdot nh + (nh)^2\} : \Sigma_1^n \{a \cdot rh + (rh)^2\} \end{aligned}$$

From these relations it is inferred that

$$(\text{frustum } BF) : (\text{volume of segment}) = (a + nh) : (\tfrac{1}{2}a + \tfrac{1}{2}b)$$

$$\text{or } (\text{volume of segment}) : (\text{volume of cone } ABB')$$

$$= (AD + 3CA) : (AD + 3CB)$$

and this is confirmed by the method of exhaustion.

The result obtained by Archimedes is equivalent to proving that, if h be indefinitely diminished while n is indefinitely increased but nh remains always equal to b , then

$$\text{limit of } n(ab + b^2) / S_n = (a + b) / (\tfrac{1}{2}a + \tfrac{1}{2}b),$$

$$hS_n = ah(h + 2h + \dots + nh) + h\{h^2 + (2h)^2 + \dots + (nh)^2\}.$$

The limit of this latter expression is what we should write

$$\int_0^b (ax + x^2) dx = b^2(\frac{1}{2}a + \frac{1}{3}b),$$

and Archimedes's procedure is the equivalent of this integration.

III. In the case of the *spheroid* (Props. 29, 30) we take a segment less than half the spheroid.

As in the case of the hyperboloid,

$$\begin{aligned} (\text{frustum in } BF) : (\text{frustum on base } QQ') \\ &= BD^2 : QM^2 \\ &= AD \cdot A'D : AM \cdot A'M; \end{aligned}$$

but, in order to reduce the summation to the same as that in Prop. 2, Archimedes expresses $AM \cdot A'M$ in a different form equivalent to the following.

Let $AD (= b)$ be divided into n equal parts of length h , and suppose that $AA' = a$, $CD = \frac{1}{2}c$.

$$\text{Then} \quad AD \cdot A'D = \frac{1}{4}a^2 - \frac{1}{4}c^2,$$

$$\begin{aligned} \text{and} \quad AM \cdot A'M &= \frac{1}{4}a^2 - (\frac{1}{2}c + rh)^2 \quad (DM = rh) \\ &= AD \cdot A'D - \{c \cdot rh + (rh)^2\} \\ &= cb + b^2 - \{c \cdot rh + (rh)^2\}. \end{aligned}$$

Thus in this case we have

(frustum BF) : (inscribed figure)

$$= n(cb + b^2) : [n(cb + b^2) - \Sigma_1^n \{c \cdot rh + (rh)^2\}]$$

and

(frustum BF) : (circumscribed figure)

$$= n(cb + b^2) : [n(cb + b^2) - \Sigma_1^{n-1} \{c \cdot rh + (rh)^2\}].$$

And, since $b = nh$, we have, by means of Prop. 2,

$$\begin{aligned} n(cb + b^2) : [n(cb + b^2) - \Sigma_1^n \{c \cdot rh + (rh)^2\}] \\ &> (c + b) : \{c + b - (\frac{1}{2}c + \frac{1}{3}b)\} \\ &> n(cb + b^2) : [n(cb + b^2) - \Sigma_1^{n-1} \{c \cdot rh + (rh)^2\}]. \end{aligned}$$

The conclusion, confirmed as usual by the method of exhaustion, is that

$$\begin{aligned} (\text{frustum } BF) : (\text{segment of spheroid}) &= (c+b) : \{c+b - (\tfrac{1}{2}c + \tfrac{2}{3}b)\} \\ &= (c+b) : (\tfrac{1}{2}c + \tfrac{2}{3}b), \end{aligned}$$

whence (volume of segment) : (volume of cone ABB')

$$\begin{aligned} &= (\tfrac{3}{2}c + 2b) : (c+b) \\ &= (3CA - AD) : (2CA - AD), \text{ since } CA = \tfrac{1}{2}c + b \end{aligned}$$

As a particular case (Props. 27, 28), half the spheroid is double of the corresponding cone.

Props. 31, 32, concluding the treatise, deduce the simple formula for the volume of the greater segment, namely, in figure,

$$\begin{aligned} (\text{greater segmt.}) : (\text{cone or segmt. of cone with same base and altitude}) \\ = (CA + AD) : AD. \end{aligned}$$

On Spirals.

The treatise *On Spirals* begins with a preface addressed to Dositheus in which Archimedes mentions the death of Conon as a grievous loss to mathematics, and then summarizes the main results of the treatises *On the Sphere and Cylinder* and *On Conoids and Spheroids*, observing that the last two propositions of Book II of the former treatise took the place of two which, as originally enunciated to Dositheus, were wrong; lastly, he states the main results of the treatise *On Spirals*, premising the definition of a spiral which is as follows:

'If a straight line one extremity of which remains fixed is made to revolve at a uniform rate in a plane until it returns to the position from which it started, and if, at the same time as the straight line is revolving, a point move at a uniform rate in a straight line from the fixed extremity to the other extremity, the path described by the point is called a spiral.'

positions (5-9) which are all of one type. Prop. 5 states that, given a circle with centre O , a tangent to it at A , and c , the

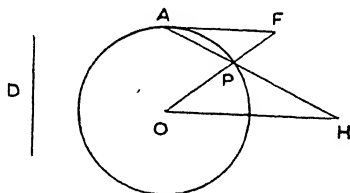


FIG. 1.

circumference of any circle whatever, it is possible to draw a straight line OPF meeting the circle in P and the tangent in F such that

$$FP : OP < (\text{arc } AP) : c.$$

Archimedes takes D a straight line greater than c , draws OH parallel to the tangent at A and then says 'let PH be placed equal to D verging ($\nu\epsilon\upsilon\upsilon\sigma\alpha$) towards A '. This is the usual phraseology of the type of problem known as $\nu\epsilon\upsilon\sigma\iota\varsigma$ where a straight line of given length has to be placed between two lines or curves in such a position that, if produced, it passes through a given point (this is the meaning of *verging*). Each of the propositions 5-9 depends on a $\nu\epsilon\upsilon\sigma\iota\varsigma$ of this kind,

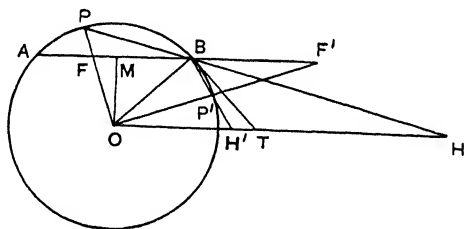


FIG. 2.

which Archimedes assumes as 'possible' without showing how it is effected. Except in the case of Prop. 5, the theoretical solution cannot be effected by means of the straight line and circle; it depends in general on the solution of an equation of the fourth degree, which can be solved by means of the

points of intersection of a certain rectangle and a certain parabola. It is quite possible, however, that such problems were in practice often solved by a mechanical method, namely by placing a ruler, by trial, in the position of the required line: for it is only necessary to place the ruler so that it passes through the given point and to turn it round that point as a pivot till the intercept between the ruler and the circle is of the given length. In Props. 6-9 we have a circle with a chord AB less than the diameter in it, OM the perpendicular from O on AB , BT the tangent at B , OT the line through O parallel to AB ; $D:E$ is any ratio less than the case may be, than the ratio $BM:MO$. (Fig. 2) show that it is possible to draw a straight

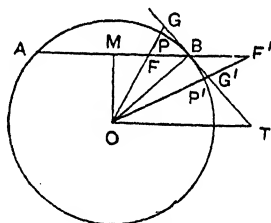


FIG. 3.

meeting AB in F and the circle in P such that $FP:BG = D:E$ (OP meeting AB in the case where $D:E < BM:MO$), and meeting AB produced when $D:E > BM:MO$).

(Fig. 3) it is proved that it is possible to draw a straight line $OFFP$ meeting AB in F , the circle in P and the tangent at B in G , such that $FP:BG = D:E$ (OP meeting AB itself in the case where $D:E < BM:MO$, and meeting AB produced in the case where $D:E > BM:MO$).

We will illustrate by the constructions in Proposition 7, as it is these propositions which are actually the basis of the method. Prop. 7. If $D:E$ is any ratio $> BM:MO$, it is required to draw $OP'F'$ meeting the circle in P' and AB produced in F' so that

$OFFG$ is the straight line required.

For $CG \cdot GT = OG \cdot GQ = OG \cdot BK$.

But $OF : OG = BT : GT$, by parallels,

whence $OF \cdot GT = OG \cdot BT$.

Therefore $CG \cdot GT : OF \cdot GT = OG \cdot BK : OG \cdot BT$

whence $CG : OF = BK : BT$

$$= BC : OB$$

$$= BC : OP.$$

Therefore $OP : OF = BC : CG$,

and hence $PF : OP = BG : BC$,

or $PF : BG = OB : BC = D : E$.

Pappus objects to Archimedes's use of the $\nu\epsilon\upsilon\sigma\iota\varsigma$ as Prop. 8, 9 in these words:

'it seems to be a grave error into which geometers fall whenever any one discovers the solution of a plane problem by means of conics or linear (higher) curves, or solves it by means of a foreign kind, as is the case e.g. with the problem in the fifth Book of the Conics of Archimedes relating to the parabola, and (2) when Archimedes assumes in his work on the spiral a $\nu\epsilon\upsilon\sigma\iota\varsigma$ of a "solid" character without reference to a circle; for it is possible without calling to aid of anything solid to find the proof of the theorem of Archimedes, that is, to prove that the circumference of a circle arrived at in the first revolution is equal to the line drawn at right angles to the initial line to meet the spiral to the spiral (i.e. the subtangent).'

There is, however, this excuse for Archimedes, that he assumes that the problem *can* be solved and does not find the actual solution. Pappus¹ himself gives a solution of the particular $\nu\epsilon\upsilon\sigma\iota\varsigma$ by means of conics. Apollonius works in the Books of $\nu\epsilon\upsilon\sigma\epsilon\iota\varsigma$, and it is quite possible that by Archimedes' time there may already have been a collection of such problems to which tacit reference was permissible.

Prop. 10 repeats the result of the Lemma to Prop.

¹ Pappus, iv, pp. 298-302.

summation, namely that

$$\begin{aligned}(n-1)(na)^2 &: \{a^2 + (2a)^2 + (3a)^2 + \dots + (\overline{n-1}a)^2\} \\ &> (na)^2 : \{na \cdot a + \tfrac{1}{3}(na-a)^2\} \\ &> (n-1)(na)^2 : \{(2a)^2 + (3a)^2 + \dots + (na)^2\}.\end{aligned}$$

The same proposition is also true if the terms of the series are a^2 , $(a+b)^2$, $(a+2b)^2$... $(a+\overline{n-1}b)^2$, and it is assumed in the more general form in Props. 25, 26.

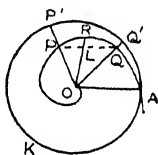
Archimedes now introduces his Definitions, of the *spiral* itself, the *origin*, the *initial line*, the *first distance* (= the radius vector at the end of one revolution), the *second distance* (= the equal length added to the radius vector during the second complete revolution), and so on; the *first area* (the area bounded by the spiral described in the first revolution and the 'first distance'), the *second area* (that bounded by the spiral described in the second revolution and the 'second distance'), and so on; the *first circle* (the circle with the 'first distance' as radius), the *second circle* (the circle with radius equal to the sum of the 'first' and 'second distances', or twice the first distance), and so on.

Props. 12, 14, 15 give the fundamental property of the spiral connecting the length of the radius vector with the angle through which the initial line has revolved from its original position, and corresponding to the equation in polar coordinates $r = a\theta$. As Archimedes does not speak of angles greater than π , or 2π , he has, in the case of points on any turn after the first, to use multiples of the circumference of a circle as well as arcs of it. He uses the 'first circle' for this purpose. Thus, if P , Q are two points on the first turn,

$$OP : OQ = (\text{arc } AKP') : (\text{arc } AKQ');$$

if P , Q are points on the n th turn of the spiral, and c is the circumference of the first circle,

$$OP : OQ = \{(n-1)c + \text{arc } AKP'\} : \{(n-1)c + \text{arc } AKQ'\}.$$



Prop. 13 proves that, if a straight line touches the spiral, it

at P touch the spiral at another point Q . Then, the angle POQ by OL meeting PQ in L and the $OP + OQ = 2OR$ by the property of the spiral and the property of the triangle (assumed, but each $OP + OQ > 2OL$, so that $OL < OR$, and some part lies within the spiral. Hence PQ cuts the spiral contrary to the hypothesis.

Props. 16, 17 prove that the angle made by the tangent at a point with the radius vector to that point is obtuse on the 'backward' side, and acute on the 'forward' side, of the radius vector.

Props. 18-20 give the fundamental propositions concerning the tangent, that is to say, they give the length of the tangent at any point P (the distance between O and the point of intersection of the tangent with the perpendicular from O to the tangent). Archimedes always deals first with the first turn of the spiral, with any subsequent turn, and with each complete turn or parts or points of any particular turn. Thus he considers the tangents in this order, (1) the tangent at A the end of the first turn, (2) the tangent at the end of the second and any subsequent turn, (3) the tangent at any intermediate point of the first or any subsequent turn. We will take as our case the case of the tangent at any intermediate point P of any turn (Prop. 20).

If OA be the initial line, P any point on the spiral, the tangent at P and OT perpendicular to OP , then it has been proved that, if ASP be the circle through P with center A meeting PT in S , then

$$(\text{subtangent } OT) = (\text{arc } ASP).$$

- I. If possible, let OT be greater than the arc ASP . Measure off OU such that $OU > \text{arc } ASP$ but $OU < OT$. Then the ratio $PO : OU$ is greater than the ratio of PO to the perpendicular on PS , i.e. greater than the ratio of $\frac{1}{2}PS$ to the perpendicular on PS .

Therefore (Prop. 7) we can draw a straight line OT perpendicular to OP such that TP produced in F , and the circle in Q , such that

$$FQ : PQ = PO : OU.$$

Then, since $PO = RO$, we have, *alternando*,

$$F'R : RO = GP : OV$$

$$> (\text{arc } PR) : (\text{arc } ASP), \text{ a fortiori,}$$

$$\begin{aligned} \text{whence } F'O : RO &< (\text{arc } ASR) : (\text{arc } ASP) \\ &< OR' : OP, \end{aligned}$$

so that $F'O < OR'$; which is impossible.

Therefore OT is not less than the arc ASP . A proved not greater than the same arc. Therefore

$$OT = (\text{arc } ASP).$$

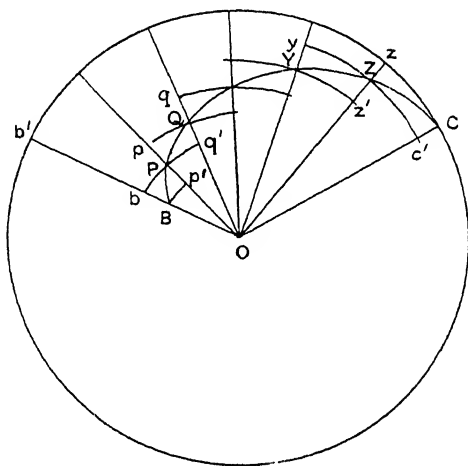
As particular cases (separately proved by Archimedes) let P be the extremity of the first turn and c_1 the circumference of the first circle, the subtangent $= c_1$; if P be the extremity of the second turn and c_2 the circumference of the second circle, the subtangent $= 2c_2$; and generally, if P be the extremity of the n th turn and c_n the circumference of the n th circle (the circle with O as centre and OP as radius), the subtangent to the tangent at the extremity of the n th turn as radius tangent to the tangent at the extremity of the n th turn is $= nc_n$.

If P is a point on the n th turn, not the extremity of the n th turn, let O as centre and OP as radius cuts the circle in K , while p is the circumference of the circle with O as centre and OP as radius tangent to the tangent at $P = (n-1)p + \text{arc } KP$ ('forward').¹

The remainder of the book (Props. 21-8) is devoted to finding the areas of portions of the spiral and of the turns cut off by the initial line or any two radii. We will illustrate by the general case (Prop. 21). Let OB, OC , two bounding radii vectores, including the initial line of the spiral. With centre O and radius OC describe a circle. Divide the angle BOC into any number of equal angles by radii OP, OQ, \dots, OY, OZ such that the radii vectores $OB, OP, OQ, \dots, OY, OZ$

¹ On the whole course of Archimedes's proof of the proposition, see note in the Appendix.

scribed to the spiral and consisting of the sum of small sectors of circles, and an inscribed figure of the same kind. As the first sector in the circumscribed figure is equal to the second sector in the inscribed, it is easily seen that the areas of the circumscribed and inscribed figures differ by the difference between the sectors OzC and OBp' ; therefore, by increasing the number of divisions of the angle BOC , we can make the



difference between the areas of the circumscribed and inscribed figures as small as we please; we have, therefore, the elements necessary for the application of the method of exhaustion.

If there are n radii $OB, OP \dots OC$, there are $(n-1)$ parts of the angle BOC . Since the angles of all the small sectors are equal, the sectors are as the square on their radii.

Thus (whole sector $OB'C$) : (circumscribed figure)

$$= (n-1)OC^2 : (OP^2 + OQ^2 + \dots + OC^2),$$

and (whole sector $OB'C$) : (inscribed figure)

$$= (n-1)OC^2 : (OB^2 + OP^2 + OQ^2 + \dots + OZ^2).$$

of n terms; therefore (cf. Prop. 11 and Cor.),

$$(n-1)OC^2 : (OP^2 + OQ^2 + \dots + OC^2)$$

$$< OC^2 : \{OC \cdot OB + \frac{1}{3}(OC - OB)^2\}$$

$$< (n-1)OC^2 : (OB^2 + OP^2 -$$

Compressing the circumscribed and inscribed figures in the usual way, Archimedes proves by exhaustion

$$(\text{sector } Ob'C) : (\text{area of spiral } OBC)$$

$$= OC^2 : \{OC \cdot OB + \frac{1}{3}(OC -$$

If $OB = b$, $OC = c$, and $(c - b) = (n - 1)h$, the result is the equivalent of saying that, when h decreases as n increases indefinitely, while $c - b$ remains constant,

$$\begin{aligned} \text{limit of } h \{b^2 + (b + h)^2 + (b + 2h)^2 + \dots + (b + (n-1)h)^2\} \\ = (c - b) \{cb + \frac{1}{3}(c^3 - b^3)\} \\ = \frac{1}{3}(c^3 - b^3); \end{aligned}$$

that is, with our notation,

$$\int_b^c x^2 dx = \frac{1}{3}(c^3 - b^3).$$

In particular, the area included by the first initial line is bounded by the radii vectores OA and OB ; the area, therefore, is to the circle with radius $2\pi a$ as 1 to $(2\pi a)^2$, that is to say, it is $\frac{1}{3}$ of the circle. This is separately proved in Prop. 24 by means of Cor. 1, 2.

The area of the ring added while the radius vector describes the second turn is the area bounded by the radii OB and $4\pi a$, and is to the circle with radius $4\pi a$ as $\{r_2 r_1 + \frac{1}{3}(r_2 - r_1)^2\}$ to r_2^2 , where $r_1 = 2\pi a$ and the ratio is $7 : 12$ (Prop. 25).

If R_1 be the area of the first turn of the spiral, R_2 the area of the ring added by the second complete turn, R_3 that of the ring added by the third, and so on, then (Prop. 27)

$$R_3 = 2R_2, \quad R_4 = 3R_2, \quad R_5 = 4R_2, \dots R_n =$$

$$\text{Also } R_2 = 6R_1.$$

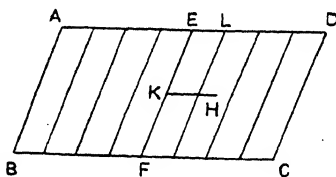
ED , and the arc ED of the circle and the arc ED of the spiral, and F the portion cut off between the arc BC of the spiral, the radius OC and the arc intercepted between OB and OC of the circle with centre O and radius OB , it is proved that

$$E : F = \{OB + \frac{2}{3}(OC - OB)\} : \{OB + \frac{1}{3}(OC - OB)\} \text{ (Prop. 28).}$$

On Plane Equilibriums, I, II.

In this treatise we have the fundamental principles of mechanics established by the methods of geometry in its strictest sense. There were doubtless earlier treatises on mechanics, but it may be assumed that none of them had been worked out with such geometrical rigour. Archimedes begins with seven Postulates including the following principles. Equal weights at equal distances balance; if unequal weights operate at equal distances, the larger weighs down the smaller. If when equal weights are in equilibrium something be added to, or subtracted from, one of them, equilibrium is not maintained but the weight which is increased or is not diminished prevails. When equal and similar plane figures coincide if applied to one another, their centres of gravity similarly coincide; and in figures which are unequal but similar the centres of gravity will be 'similarly situated'. In any figure the contour of which is concave in one and the same direction the centre of gravity must be within the figure. Simple propositions (1-5) follow, deduced by *reductio ad absurdum*; these lead to the fundamental theorem, proved first for commensurable and then by *reductio ad absurdum* for incommensurable magnitudes, that *Two magnitudes, whether commensurable or incommensurable, balance at distances reciprocally proportional to the magnitudes* (Props. 6, 7). Prop. 8 shows how to find the centre of gravity of a part of a magnitude when the centres of gravity of the other part and of the whole magnitude are given. Archimedes then addresses himself to the main problems of Book I, namely to find the centres of gravity of (1) a parallelogram (Props. 9, 10), (2) a triangle (Props. 13, 14), and (3) a parallelogram (Prop. 15), and here we have an illustration of the extraordinary rigour which he requires in his geometrical

that, if all the lines that can be drawn in a figure parallel to one side have their middle points in a straight line, the centre of gravity must lie somewhere on that line; he is not content to regard the figure as made up of an infinity of such parallel lines; pure geometry realises that the parallelogram is made up of elementary parallelograms indefinitely narrow if you please, but still parallelograms; the triangle of elementary *trapezia*, not straight lines; that to assume directly that the centre of gravity lies on a straight line bisecting the parallelograms would be a *petitio principii*. Accordingly the result, not yet covered in the informal way, is clinched by a proof by *ad absurdum* in each case. In the case of the parallelogram $ABCD$ (Prop. 9), if the centre of gravity is not on the line EF bisecting two opposite sides, let it be at H . Draw HK parallel to AD . Then it is possible by bisecting AK , then bisecting the halves, and so on, ultimately to find a length less than KH . Let this be done, and through



points of division of AD draw parallels to AB or DC to divide the parallelogram into a number of equal and similar parallelograms as in the diagram. The centre of gravity of each of these parallelograms lies on a straight line similarly situated with regard to it. Hence we have a series of centres of gravity of equal magnitudes with their centres of gravity at equal distances along a straight line. Therefore the centre of gravity of the whole is on the line joining the centres of gravity of the two middle parallelograms (Prop. 5, Cor. 2). But this is impossible, because H is outside those parallelograms. Therefore the centre of gravity cannot but lie on EF .

Similarly the centre of gravity lies on the straight line bisecting the other opposite sides AB, CD ; therefore the centre of gravity is at the intersection of this line with EF , i.e. at the intersection of the diagonals.

Let AD be the median through A . The centre of gravity must lie on AD .

For, if not, let it be at H , and draw HI parallel to BC . Then, if we bisect DC , then bisect the halves, and so on, we shall arrive at a length DE less than IH . Divide BC into lengths equal to DE , draw parallels to DA through the points of division, and complete the small parallelograms as shown in the figure.

The centres of gravity of the whole parallelograms SN , TP , FQ lie on AD (Prop. 9); therefore the centre of gravity of the

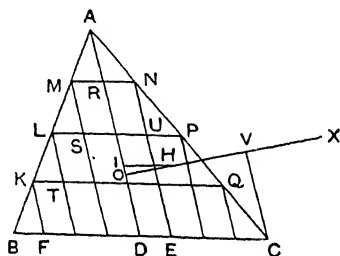


figure formed by them all lies on AD ; let it be O . Join OH , and produce it to meet in V the parallel through C to AD .

Now it is easy to see that, if n be the number of parts into which DC , AC are divided respectively,

$$\begin{aligned} (\text{sum of small } \Delta s \text{ } AMR, MLS \dots ARN, NUP \dots) : (\Delta ABC) \\ &= n \cdot AN^2 : AC^2 \\ &= 1 : n ; \end{aligned}$$

whence

$$(\text{sum of small } \Delta s) : (\text{sum of parallelograms}) = 1 : (n-1).$$

Therefore the centre of gravity of the figure made up of all the small triangles is at a point X on OH produced such that

$$XH = (n-1)OH.$$

But $VH:HO < CE:ED$ or $(n-1):1$; therefore $XH > VH$.

It follows that the centre of gravity of all the small triangles taken together lies at X notwithstanding that all the triangles lie on one side of the parallel to AD drawn through X : which is impossible.

Hence the centre of gravity of the whole triangle can but lie on AD .

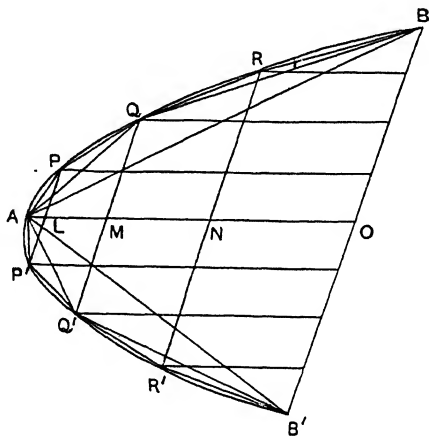
It lies, similarly, on either of the other two medians that it is at the intersection of any two medians (Prop. 14).

Archimedes gives alternative proofs of a direct character both for the parallelogram and the triangle, depending on postulate that the centres of gravity of similar figures 'similarly situated' in regard to them (Prop. 10 for parallelogram, Props. 11, 12 and part 2 of Prop. 13 for triangle).

The geometry of Prop. 15 deducing the centre of gravity of a trapezium is also interesting. It is proved that, if AD , BC are the parallel sides (AD being the smaller), and EF is a straight line joining their middle points, the centre of gravity is at a point G on EF such that

$$GE : GF = (2BC + AD) : (2AD + BC).$$

Book II of the treatise is entirely devoted to finding centres of gravity of a parabolic segment (Props. 1-8) of a portion of it cut off by a parallel to the base (Props. 9, 10). Prop. 1 (really a particular case of I. 6, 7) proves that, if AB

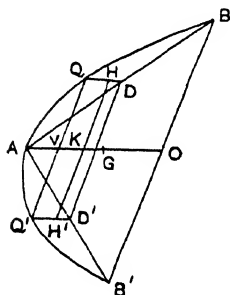


the course of which is characteristic and deserves to be set out. Archimedes uses a series of figures inscribed to the segment, as he says, 'in the recognized manner' (*γνωρίμως*). The rule is as follows. Inscribe in the segment the triangle ABB' with the same base and height; the vertex A is then the point of contact of the tangent parallel to BB' . Do the same with the remaining segments cut off by AB , AB' , then with the segments remaining, and so on. If $BRQPAP'Q'R'B'$ is such a figure, the diameters through Q , Q' , P , P' , R , R' bisect the straight lines AB , AB' , AQ , AQ' , QB , $Q'B'$ respectively, and BB' is divided by the diameters into parts which are all equal. It is easy to prove also that PP' , QQ' , RR' are all parallel to BB' , and that $AL:LM:MN:NO = 1:3:5:7$, the same relation holding if the number of sides of the polygon is increased; i.e. the segments of AO are always in the ratio of the successive odd numbers (Lemmas to Prop. 2). The centre of gravity of the inscribed figure lies on AO (Prop. 2). If there be two parabolic segments, and two figures inscribed in them 'in the recognized manner' with an equal number of sides, the centres of gravity divide the respective axes in the same proportion, for the ratio depends on the same ratio of odd numbers $1:3:5:7 \dots$ (Prop. 3). The centre of gravity of the parabolic segment itself lies on the diameter AO (this is proved in Prop. 4 by *reductio ad absurdum* in exactly the same way as for the triangle in I. 13). It is next proved (Prop. 5) that the centre of gravity of the segment is nearer to the vertex A than the centre of gravity of the inscribed figure is; but that it is possible to inscribe in the segment in the recognized manner a figure such that the distance between the centres of gravity of the segment and of the inscribed figure is less than any assigned length, for we have only to increase the number of sides sufficiently (Prop. 6). Incidentally, it is observed in Prop. 4 that, if in any segment the triangle with the same base and equal height is inscribed, the triangle is greater than half the segment, whence it follows that, each time we increase the number of sides in the inscribed figure, we take away more than half of the segments remaining over; and in Prop. 5 that corresponding segments on opposite sides of the axis, e.g. QRB , $Q'R'B'$ have their axes equal and therefore are equal in

their centres of gravity divide their diameters in the same ratio (Archimedes enunciates this of similar segments, but it is true of any two segments and is required for the proof of Prop. 8). Prop. 8 now finds the centre of gravity of any segment by using the last proposition and the geometrical equivalent of the solution of a simple problem, namely the ratio (m , say) of AG to AO , where G is the centre of gravity of the segment.

Since the segment $ABB' = \frac{4}{3}(\Delta ABB')$, the sum of the moments of the segments $AQB, AQ'B' = \frac{1}{3}(\Delta ABB')$.

Further, if $QD, Q'D'$ are the diameters of the two segments,



$QD, Q'D'$ are equal, and, since the centres of gravity H, H' of the segments are on $QD, Q'D'$ proportionally, HH' is parallel to QQ' , and the centre of gravity of the two segments together is at G on AO , where HH' meets AO .

Now $AO = 4AV$ (Lemma 2), and $QD = \frac{1}{2}AO - AV$. H divides QD in the same ratio as V divides AO (Prop. 7); therefore

$$VK = QH = m \cdot QD =$$

Taking moments about A of the segment, the triangle ABB' , and the sum of the small segments, we have (dividing by AV and $\Delta ABB'$)

$$\frac{1}{3}(1+m) + \frac{2}{3} \cdot 4 = \frac{4}{3} \cdot 4m,$$

or

$$15m = 9,$$

and $m = \frac{3}{5}$.

That is, $AG = \frac{3}{5}AO$, or $AG:GO = 3:2$.

The final proposition (10) finds the centre of gravity of a portion of a parabola cut off between two parallel lines BB' . If PP' is the shorter of the chords and BB' is the longer, bisecting PP' , BB' meets them in N, O respectively. Archimedes proves that, if NO be divided into five equal parts, the centre of gravity is at M , which LM is the middle one (L being nearer to N).

$$LG:GM = BO^2.(2PN+BO):PN^2.(2BO+PN).$$

The geometrical proof is somewhat difficult, and uses a very remarkable Lemma which forms Prop. 9. If a, b, c, d, x, y are straight lines satisfying the conditions

$$\left. \begin{aligned} \frac{a}{b} = \frac{b}{c} = \frac{c}{d} \quad (a > b > c > d), \\ \frac{d}{a-d} = \frac{x}{\frac{3}{5}(a-c)}, \\ \frac{2a+4b+6c+3d}{5a+10b+10c+5d} = \frac{y}{a-c}, \end{aligned} \right\}$$

and

then must

$$x+y = \frac{2}{5}a.$$

The proof is entirely geometrical, but amounts of course to the elimination of three quantities b, c, d from the above four equations.

The Sand-reckoner (*Psammites* or *Arenarius*).

I have already described in a previous chapter the remarkable system, explained in this treatise and in a lost work, *Ἀρχαί*, *Principles*, addressed to Zeuxippus, for expressing very large numbers which were beyond the range of the ordinary Greek arithmetical notation. Archimedes showed that his system would enable any number to be expressed up to that which in our notation would require 80,000 million million ciphers and then proceeded to prove that this system more than sufficed to express the number of grains of sand which it would take to fill the universe, on a reasonable view (as it seemed to him) of the size to be attributed to the universe. Interesting as the book is for the course of the argument by which Archimedes establishes this, it is, in addition, a document of the first importance historically. It is here that we learn that Aristarchus put forward the Copernican theory of the universe, with the sun in the centre and the planets including the earth revolving round it, and that Aristarchus further discovered the angular diameter of the sun to be $\frac{1}{20}$ th of the circle of the zodiac or half a degree. Since Archimedes, in order to calculate a safe figure (not too small) for the size

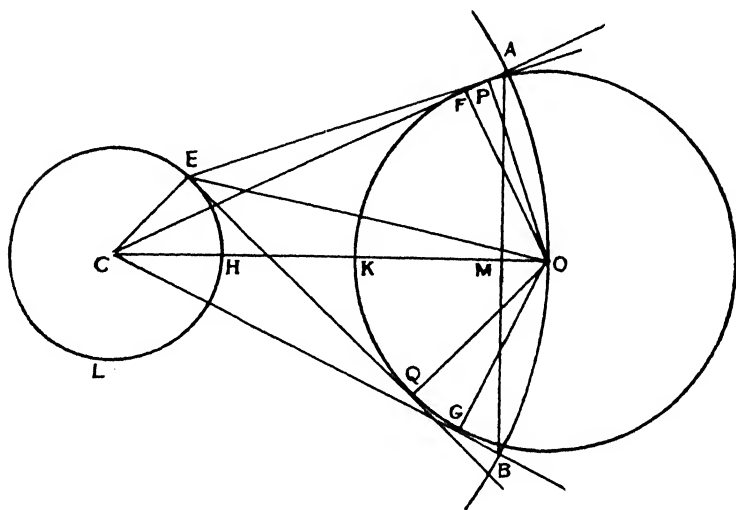
of the universe, has to make certain assumptions as to the sizes and distances of the sun and moon and their relation to the size of the universe, he takes the opportunity of quoting earlier views. Some have tried, he says, to prove that the perimeter of the earth is about 300,000 stades; in order to be quite safe he will take it to be about ten times this, or 3,000,000 stades, and not greater. The diameter of the earth, like most earlier astronomers, he takes to be greater than that of the moon but less than that of the sun. Eudoxus, he says, declared the diameter of the sun to be ninety times that of the moon, Phidias, his own father, twelve times, while Aristarchus tried to prove that it is greater than 18 but less than 20 times the diameter of the moon; he will again on the safe side and take it to be 30 times, but not more. This position is rather more difficult as regards the ratio of the distance of the sun to the size of the universe. Here he seizes upon a dictum of Aristarchus that the sphere of the fixed stars is so great that the circle in which he supposes the earth to revolve (round the sun) 'bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface'. If this is taken in a strictly mathematical sense it means that the sphere of the fixed stars is infinite in size which would not suit Archimedes's purpose; to get another meaning out of it he presses the point that Aristarchus's words cannot be taken quite literally because the centre, being without magnitude, cannot be in any ratio to any other magnitude; hence he suggests that a reasonable interpretation of the statement would be to suppose that, if we conceive a sphere with radius equal to the distance between the centre of the sun and the centre of the earth, then

$$(\text{diam. of earth}) : (\text{diam. of said sphere})$$

$$= (\text{diam. of said sphere}) : (\text{diam. of sphere of fixed stars})$$

This is, of course, an arbitrary interpretation; Aristarchus presumably meant no such thing, but merely that the size of the earth is negligible in comparison with that of the sphere

with the circumference of the circle described by its centre. Aristarchus had made the apparent diameter of the sun $\frac{1}{720}$ th of the said circumference; Archimedes will prove that the said circumference cannot contain as many as 1,000 sun's diameters, or that the diameter of the sun is greater than the side of a regular chiliagon inscribed in the circle. First he made an experiment of his own to determine the apparent diameter of the sun. With a small cylinder or disc in a plane at right angles to a long straight stick and moveable along it, he observed the sun at the moment when it cleared the horizon in rising, moving the disc till it just covered and just failed to cover the sun as he looked along the straight stick. He thus found the angular diameter to lie between $\frac{1}{164}R$ and $\frac{1}{200}R$, where R is a right angle. But as, under his assumptions, the size of the earth is not negligible in comparison with the sun's circle, he had to allow for parallax and find limits for the angle subtended by the sun at the centre of the earth. This he does by a geometrical argument very much in the manner of Aristarchus.



Let the circles with centres O, C represent sections of the sun and earth respectively, E the position of the observer observing

the sun when it has just cleared the horizon. Draw two tangents EP , EQ to the circle with centre O , and from C let CF , CG be drawn touching the same circle. With C and radius CO describe a circle: this will represent the orbit of the centre of the sun round the earth. Let this circle be tangent to the tangents from C in A , B , and join AB meeting CO in H .

Archimedes's observation has shown that

$$\frac{1}{164}R > \angle PEQ > \frac{1}{200}R;$$

and he proceeds to prove that AB is less than the side of a regular polygon of 656 sides inscribed in the circle, but greater than the side of an inscribed regular polygon of 1,000 sides, in other words, that

$$\frac{1}{164}R > \angle FCG > \frac{1}{250}R.$$

The first relation is obvious, for, since $CO > EO$,

$$\angle PEQ > \angle FCG.$$

Next, the perimeter of any polygon inscribed in the circle AOB is less than $\frac{44}{7}CO$ (i.e. $\frac{22}{7}$ times the diameter);

Therefore $AB < \frac{1}{656} \cdot \frac{44}{7}CO$ or $\frac{11}{1148}CO$,

and, *a fortiori*, $AB < \frac{1}{100}CO$.

Now, the triangles CAM , COF being equal in all respects, $AM = OF$, so that $AB = 2OF = (\text{diameter of sun}) > CH$, since the diameter of the sun is greater than that of the earth. Therefore $CH + OK < \frac{1}{100}CO$, and $HK > \frac{99}{100}CO$.

And $CO > CF$, while $HK < EQ$, so that $EQ > \frac{99}{100}CO$.

We can now compare the angles OCF , OEQ ;

$$\begin{aligned} \text{for } \frac{\angle OCF}{\angle OEQ} & \left[> \frac{\tan OCF}{\tan OEQ} \right] \\ & > \frac{EQ}{CF} \\ & > \frac{99}{100}, \text{ a fortiori.} \end{aligned}$$

Doubling the angles, we have

$$\begin{aligned} \angle FCG & > \frac{99}{100} \cdot \angle PEQ \\ & > \frac{99}{20000}R, \text{ since } \angle PEQ > \frac{1}{200}R, \\ & > \frac{1}{203}R. \end{aligned}$$

612 sides, and a yet more greater than the side of a regular polygon of 1,000 sides, inscribed in the circle AOB .

The perimeter of the chiliagon, as of any regular polygon with more sides than six, inscribed in the circle AOB is greater than 3 times the diameter of the sun's orbit, but is less than 1,000 times the diameter of the sun, and *a fortiori* less than 30,000 times the diameter of the earth;

therefore (diameter of sun's orbit) $< 10,000$ (diam. of earth)
 $< 10,000,000,000$ stades.

But (diam. of earth) : (diam. of sun's orbit)

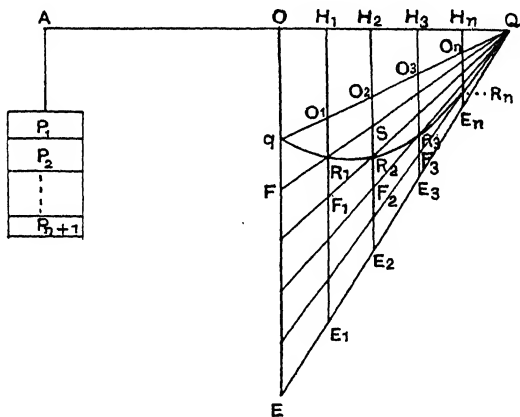
= (diam. of sun's orbit) : (diam. of universe);

therefore the universe, or the sphere of the fixed stars, is less than $10,000^3$ times the sphere in which the sun's orbit is a great circle.

Archimedes takes a quantity of sand not greater than a poppy-seed and assumes that it contains not more than 10,000 grains; the diameter of a poppy-seed he takes to be not less than $\frac{1}{40}$ th of a finger-breadth; thus a sphere of diameter 1 finger-breadth is not greater than 64,000 poppy-seeds and therefore contains not more than 640,000,000 grains of sand ('6 units of *second order* + 40,000,000 units of *first order*') and *a fortiori* not more than 1,000,000,000 ('10 units of *second order* of numbers'). Gradually increasing the diameter of the sphere by multiplying it each time by 100 (making the sphere 1,000,000 times larger each time) and substituting for 10,000 finger-breadths a stadium ($< 10,000$ finger-breadths), he finds the number of grains of sand in a sphere of diameter 10,000,000,000 stadia to be less than '1,000 units of *seventh order* of numbers' or 10^{51} , and the number in a sphere 10,000³ times this size to be less than '10,000,000 units of the *eighth order* of numbers' or 10^{63} .

The Quadrature of the Parabola.

In the preface, addressed to Dositheus after the death of Conon, Archimedes claims originality for the solution of the problem of finding the area of a segment of a parabola cut off by any chord, which he says he first discovered by means of mechanics and then confirmed by means of geometry, using the lemma that, if there are two unequal areas (or magnitudes



Now Archimedes has proved in a series of propositions (6-13) that, if a trapezium such as $O_1E_1E_2O_2$ is suspended from H_1H_2 , and an area P suspended at A balances $O_1E_1E_2O_2$ so suspended, it will take a greater area than P suspended at A to balance the same trapezium suspended from H_2 and a less area than P to balance the same trapezium suspended from H_1 . A similar proposition holds with regard to a triangle such as E_nH_nQ suspended where it is and suspended at Q and H_n respectively.

Suppose (Props. 14, 15) the triangle QqE suspended where it is from OQ , and suppose that the trapezium EO_1 , suspended where it is, is balanced by an area P_1 suspended at A , the trapezium E_1O_2 , suspended where it is, is balanced by P_2 suspended at A , and so on, and finally the triangle E_nO_nQ , suspended where it is, is balanced by P_{n+1} suspended at A ; then $P_1 + P_2 + \dots + P_{n+1}$ at A balances the whole triangle, so that

$$P_1 + P_2 + \dots + P_{n+1} = \frac{1}{3} \Delta E q Q,$$

since the whole triangle may be regarded as suspended from the point on OQ vertically above its centre of gravity.

$$\text{Now } AO:OH_1 = QO:OH_1$$

$$= Qq:qO_1$$

$$= E_1O_1:O_1R_1, \text{ by Prop. 5,}$$

$$= (\text{trapezium } EO_1):(\text{trapezium } FO_1),$$

the trapezium EO_1 suspended at H_1 . And P_1 balances it where it is.

Therefore $(FO_1) > P_1$.

Similarly $(F_1O_2) > P_2$, and so on.

Again $AO : OH_1 = E_1O_1 : O_1R_1$
 $= (\text{trapezium } E_1O_2) : (\text{trapezium } L$

that is, (R_1O_2) at A will balance (E_1O_2) suspended

while P_2 at A balances (E_1O_2) suspended where it is,

whence $P_2 > R_1O_2$.

Therefore $(F_1O_2) > P_2 > (R_1O_2)$,

$(F_2O_3) > P_3 > R_2O_3$, and so on

and finally, $\Delta E_nO_nQ > P_{n+1} > \Delta R_nO_nQ$.

By addition,

$(R_1O_2) + (R_2O_3) + \dots + (\Delta R_nO_nQ) < P_2 + P_3 + \dots +$

therefore, *a fortiori*,

$(R_1O_2) + (R_2O_3) + \dots + \Delta R_nO_nQ < P_1 + P_2 + \dots + P_n$
 $< (FO_1) + (F_1O_2) + \dots + \Delta$

That is to say, we have an inscribed figure composed of trapezia and a triangle which is less, and a circumscribed figure composed in the same way which is greater, than

$P_1 + P_2 + \dots + P_{n+1}$, i.e. $\frac{1}{3} \Delta EqQ$.

It is therefore inferred, and proved by the method of exhaustion, that the segment itself is *equal* to $\frac{1}{3} \Delta EqQ$.

In order to enable the method to be applied, it is to be proved that, by increasing the number of pairs of inscribed and circumscribed figures sufficiently, the difference between the circumscribed and inscribed figures can be made as small as we please, as can be seen thus. We have first to show that all the triangles qF , into which qE is divided are equal.

We have $E_1O_1 : O_1R_1 = QO : OH_1 = (n+1) : 1$,

or $O_1R_1 = \frac{1}{n+1} \cdot E_1O_1$, whence also $O_2S = \frac{1}{n+1} \cdot E_2O_2$

or

$$O_2 R_2 = \frac{2}{n+1} \cdot O_2 E_2.$$

It follows that $O_2 S = S R_2$, and so on.

Consequently $O_1 R_1$, $O_2 R_2$, $O_3 R_3 \dots$ are divided into 1, 2, 3 ... equal parts respectively by the lines from Q meeting qE .

It follows that the difference between the circumscribed and inscribed figures is equal to the triangle FqQ , which can be made as small as we please by increasing the number of divisions in Qq , i.e. in qE .

Since the area of the segment is equal to $\frac{1}{3} \Delta E q Q$, and it is easily proved (Prop. 17) that $\Delta E q Q = 4$ (triangle with same base and equal height with segment), it follows that the area of the segment $= \frac{4}{3}$ times the latter triangle.

It is easy to see that this solution is essentially the same as that given in *The Method* (see pp. 29-30, above), only in a more orthodox form (geometrically speaking). For there Archimedes took the sum of all the *straight lines*, as $O_1 R_1$, $O_2 R_2 \dots$, as making up the segment notwithstanding that there are an infinite number of them and straight lines have no breadth. Here he takes inscribed and circumscribed trapezia proportional to the straight lines and having finite breadth, and then compresses the figures together into the segment itself by increasing indefinitely the number of trapezia in each figure, i.e. diminishing their breadth indefinitely.

The procedure is equivalent to an integration, thus:

If X denote the area of the triangle FqQ , we have, if n be the number of parts in Qq ,

(circumscribed figure)

$$\begin{aligned} &= \text{sum of } \Delta s QqF, QR_1 F_1, QR_2 F_2, \dots \\ &= \text{sum of } \Delta s QqF, QO_1 R_1, QO_2 S, \dots \\ &= X \left\{ 1 + \frac{(n-1)^2}{n^2} + \frac{(n-2)^2}{n^2} + \dots + \frac{1}{n^2} \right\} \\ &= \frac{1}{n^2 X^2} \cdot X (X^2 + 2^2 X^2 + 3^2 X^2 + \dots + n^2 X^2). \end{aligned}$$

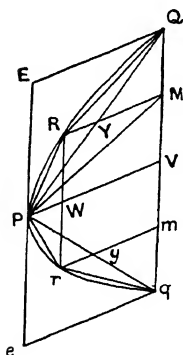
Similarly, we find that

$$(\text{inscribed figure}) = \frac{1}{n^2 X^2} \cdot X \{ X^2 + 2^2 X^2 + \dots + (n-1)^2 X^2 \}.$$

Taking the limit, we have, if A denote the area of triangle EgQ , so that $A = nX$,

$$\begin{aligned}\text{area of segment} &= \frac{1}{A^2} \int_0^A X^2 dX \\ &= \frac{1}{3} A.\end{aligned}$$

II. The purely geometrical method simply *exhausts* the parabolic segment by inscribing successive figures 'in the recognized manner' (see p. 79, above). For this purpose it is necessary to find, in terms of the triangle with the



base and height, the area added to the inscribed figure by doubling the number of sides other than the base of the segment.

Let QPq be the triangle inscribed 'in the recognized manner', P being the point of contact of the tangent parallel to Qq . PV the diameter bisecting Qq . If QV be bisected in M , m , and RM , rm be drawn parallel to PV meeting the curve in the latter points are vertices of the figure inscribed 'in the recognized manner'. For RY , ry are diameters bisecting PQ respectively.

Now $QV^2 = 4RW^2$, so that $PV = 4PW$, or $RM = 3$

But $YM = \frac{1}{2}PV = 2PW$, so that $YM = 2RY$.

Therefore $\Delta PRQ = \frac{1}{2}\Delta PQM = \frac{1}{4}\Delta PQV$.

Similarly

$\Delta Prq = \frac{1}{4}\Delta PVq$; whence $(\Delta PRQ + \Delta Prq) = \frac{1}{4}\Delta PQq$. (Prop

In like manner it can be proved that the next addition to the inscribed figure adds $\frac{1}{4}$ of the sum of $\Delta s PRQ$, and so on.

Therefore the area of the inscribed figure

$$= \left\{ 1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \dots \right\} \Delta PQq \quad (\text{Prop. 1})$$

more than half the segment. And so on (Prop. 20).

We now have to sum n terms of the above geometrical series. Archimedes enunciates the problem in the form, Given a series of areas $A, B, C, D \dots Z$, of which A is the greatest, and each is equal to four times the next in order, then (Prop. 23)

$$A + B + C + \dots + Z + \frac{1}{3}Z = \frac{4}{3}A.$$

The algebraical equivalent of this is of course

$$1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \dots + \left(\frac{1}{4}\right)^{n-1} = \frac{4}{3} - \frac{1}{3} \left(\frac{1}{4}\right)^{n-1} = \frac{1 - \left(\frac{1}{4}\right)^n}{1 - \frac{1}{4}}.$$

To find the area of the segment, Archimedes, instead of taking the limit, as we should, uses the method of *reductio ad absurdum*.

Suppose $K = \frac{4}{3} \cdot \Delta PQq$.

(1) If possible, let the area of the segment be greater than K .

We then inscribe a figure 'in the recognized manner' such that the segment exceeds it by an area less than the excess of the segment over K . Therefore the inscribed figure must be greater than K , which is impossible since

$$A + B + C + \dots + Z < \frac{4}{3}A,$$

where $A = \Delta PQq$ (Prop. 23).

(2) If possible, let the area of the segment be less than K .

If then $\Delta PQq = A$, $B = \frac{1}{4}A$, $C = \frac{1}{4}B$, and so on, until we arrive at an area X less than the excess of K over the area of the segment, we have

$$A + B + C + \dots + X + \frac{1}{3}X = \frac{4}{3}A = K.$$

Thus K exceeds $A + B + C + \dots + X$ by an area less than X , and exceeds the segment by an area greater than X .

It follows that $A + B + C + \dots + X > (\text{the segment})$; which is impossible (Prop. 22).

Therefore the area of the segment, being neither greater nor less than K , is equal to K or $\frac{4}{3}\Delta PQq$.

On Floating Bodies, I, II.

In Book I of this treatise Archimedes lays down the fundamental principles of the science of hydrostatics. These are

deduced from Postulates which are only two in number. first which begins Book I is this:

'let it be assumed that a fluid is of such a nature that, of parts of it which lie evenly and are continuous, that which is pressed the less is driven along by that which is pressed more; and each of its parts is pressed by the fluid which is perpendicular above it except when the fluid is shut up by anything and pressed by something else';

the second, placed after Prop. 7, says

'let it be assumed that, of bodies which are borne up by a fluid, each is borne upwards along the perpendicular direction through its centre of gravity'.

Prop. 1 is a preliminary proposition about a sphere, then Archimedes plunges *in medias res* with the theorem (Prop. 2) that '*the surface of any fluid at rest is a sphere the centre of which is the same as that of the earth*', and in the whole of Book I the surface of the fluid is always shown in the diagrams as spherical. The method of proof is similar to what we should expect in a modern elementary textbook, the main propositions established being the following. A solid which, size for size, is of equal weight with a fluid will, if let down into the fluid, sink till it is just covered but not lower (Prop. 3); a solid lighter than a fluid will, if let down into it, be only partly immersed, in fact just so far that the weight of the solid is equal to the weight of the fluid displaced (Props. 4, 5), and, if it is forcibly immersed, it will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced (Prop. 6).

The important proposition follows (Prop. 7) that a solid heavier than a fluid will, if placed in it, sink to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced.

crown.

Let W be the weight of the crown, w_1 and w_2 the weights of the gold and silver in it respectively, so that $W = w_1 + w_2$.

(1) Take a weight W of pure gold and weigh it in the fluid. The apparent loss of weight is then equal to the weight of the fluid displaced; this is ascertained by weighing. Let it be F_1 .

It follows that the weight of the fluid displaced by a weight w_1 of gold is $\frac{w_1}{W} \cdot F_1$.

(2) Take a weight W of silver, and perform the same operation. Let the weight of the fluid displaced be F_2 . Then the weight of the fluid displaced by a weight w_2 of silver is $\frac{w_2}{W} \cdot F_2$.

(3) Lastly weigh the crown itself in the fluid, and let F be loss of weight or the weight of the fluid displaced.

We have then
$$\frac{w_1}{W} \cdot F_1 + \frac{w_2}{W} \cdot F_2 = F,$$

that is,
$$w_1 F_1 + w_2 F_2 = (w_1 + w_2) F,$$

whence
$$\frac{w_1}{w_2} = \frac{F_2 - F}{F - F_1}.$$

According to the author of the poem *de ponderibus et mensuris* (written probably about A.D. 500) Archimedes actually used a method of this kind. We first take, says our authority, two equal weights of gold and silver respectively and weigh them against each other when both are immersed in water; this gives the relation between their weights in water, and therefore between their losses of weight in water. Next we take the mixture of gold and silver and an equal weight of silver, and weigh them against each other in water in the same way.

Nevertheless I do not think it probable that this was the way in which the solution of the problem was *discovered*. As we are told that Archimedes discovered it in his bath, and that he noticed that, if the bath was full when he entered it, so much water overflowed as was displaced by his body, he is more likely to have discovered the solution by the alternative

successively the *volumes* of fluid displaced by three equal weights, (1) the crown, (2) an equal weight of gold, (3) an equal weight of silver respectively. Suppose, as before, the weight of the crown is W and that it contains weights w_1 and w_2 of gold and silver respectively. Then

- (1) the crown displaces a certain volume of the fluid, V , say,
- (2) the weight W of gold displaces a volume V_1 , say, of fluid;

therefore a weight w_1 of gold displaces a volume $\frac{w_1}{W} \cdot V$ of the fluid;

- (3) the weight W of silver displaces V_2 , say, of the fluid;
- therefore a weight w_2 of silver displaces $\frac{w_2}{W} \cdot V_2$.

It follows that
$$V = \frac{w_1}{W} \cdot V_1 + \frac{w_2}{W} \cdot V_2,$$

whence we derive (since $W = w_1 + w_2$)

$$\frac{w_1}{w_2} = \frac{V_2 - V}{V - V_1},$$

the latter ratio being obviously equal to that obtained by the other method.

The last propositions (8 and 9) of Book I deal with the equilibrium of any segment of a sphere lighter than a fluid and immersed in it in such a way that either (1) the curved surface is downwards and the base is entirely outside the fluid, or (2) the curved surface is upwards and the base is entirely submerged, and it is proved that, in either case the segment is in stable equilibrium when the axis is vertical. This is expressed in the corresponding propositions of Book II by saying that, 'if the figure be forced into such a position that the flat surface of the segment touches the fluid (at one point), the figure will not remain inclined but will return to the upright position.'

Book II, which investigates fully the conditions of stable equilibrium of a right segment of a paraboloid of revolution floating in a fluid for different values of the specific gravity and different ratios between the axis or height of the segment and

tour de force which must be read in full to be appreciated. Prop. 1 is preliminary, to the effect that, if a solid lighter than a fluid be at rest in it, the weight of the solid will be to that of the same volume of the fluid as the immersed portion of the solid is to the whole. The results of the propositions about the segment of a paraboloid may be thus summarized. Let h be the axis or height of the segment, p the principal parameter of the generating parabola, s the ratio of the specific gravity of the solid to that of the fluid (s always < 1). The segment is supposed to be always placed so that its base is either entirely above, or entirely below, the surface of the fluid, and what Archimedes proves in each case is that, if the segment is so placed with its axis inclined to the vertical at any angle, it will not rest there but will return to the position of stability.

I. If h is not greater than $\frac{3}{4}p$, the position of stability is with the axis vertical, whether the curved surface is downwards or upwards (Props. 2, 3).

II. If h is greater than $\frac{3}{4}p$, then, in order that the position of stability may be with the axis vertical, s must be not less than $(h - \frac{3}{4}p)^2/h^2$ if the curved surface is downwards, and not greater than $\{h^2 - (h - \frac{3}{4}p)^2\}/h^2$ if the curved surface is upwards (Props. 4, 5).

III. If $h > \frac{3}{4}p$, but $h/\frac{1}{2}p < 15/4$, the segment, if placed with one point of the base touching the surface, will never remain there whether the curved surface be downwards or upwards (Props. 6, 7). (The segment will move in the direction of bringing the axis nearer to the vertical position.)

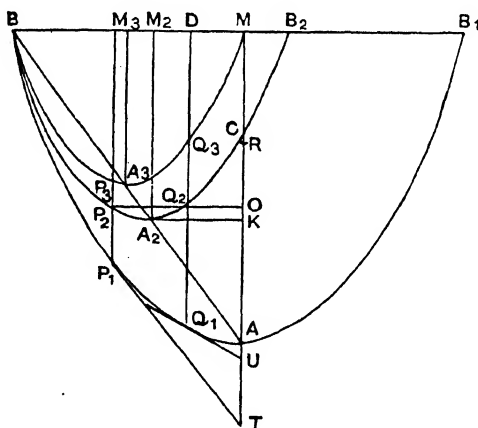
IV. If $h > \frac{3}{4}p$, but $h/\frac{1}{2}p < 15/4$, and if s is less than $(h - \frac{3}{4}p)^2/h^2$ in the case where the curved surface is downwards, but greater than $\{h^2 - (h - \frac{3}{4}p)^2\}/h^2$ in the case where the curved surface is upwards, then the position of stability is one in which the axis is not vertical but inclined to the surface of the fluid at a certain angle (Props. 8, 9). (The angle is drawn in an auxiliary figure. The construction for it in Prop. 8 is equivalent to the solution of the following equation in θ ,

$$\frac{1}{4}p \cot^2 \theta = \frac{2}{3}(h - k) - \frac{1}{2}p,$$

the surface of the fluid.)

V. Prop. 10 investigates the positions of stability in where $h/\frac{1}{2}p > 15/4$, the base is entirely above the surface. s has values lying between five pairs of ratios respectively. Only in the case where s is not less than $(h - \frac{3}{4}p)^2/h^2$ is there a position of stability that in which the axis is vertical.

BAB_1 is a section of the paraboloid through the axis. C is a point on AM such that $AC = 2CM$, K is a point on AM such that $AM:CK = 15:4$. CO is measured along the axis such that $CO = \frac{1}{2}p$, and R is a point on AM such that $MR = CO$. A_2 is the point in which the perpendicular to AM from O meets AB , and A_3 is the middle point of AB . BA_2B_2 and BA_3B_3 are parabolic segments on A_2M_2 , A_3M_3 (parallel to AM)



and similar to the original segment. (The parabola BA_2B_2 is proved to pass through C by using the above ratio $AM:CK = 15:4$ and applying Prop. 4 of the *Quadrature of the Parabola*.) The perpendicular to AM from O meets the parabola BA_2B_2 in two points P_2 , Q_2 , and straight lines through these points parallel to AM meet the other parabolas in P_1 , Q_1 and P_3 , Q_3 respectively. P_1T and P_3T are tangents to the original parabola meeting the axis at T , U . Then

(i) if s is not less than $AR^2:AM^2$ or $(h - \frac{3}{4}p)^2:h^2$, there is a stable equilibrium when AM is vertical;

only, but in a position with the base entirely out of the fluid and the axis making with the surface an angle greater than U ;

(iii a) if $s = Q_1 Q_3^2 : AM^2$, there is stable equilibrium with one point of the base touching the surface and AM inclined to it at an angle equal to U ;

(iii b) if $s = P_1 P_3^2 : AM^2$, there is stable equilibrium with one point of the base touching the surface and with AM inclined to it at an angle equal to T ;

(iv) if $s > P_1 P_3^2 : AM^2$ but $< Q_1 Q_3^2 : AM^2$, there will be stable equilibrium in a position in which the base is more submerged ;

(v) if $s < P_1 P_3^2 : AM^2$, there will be stable equilibrium with the base entirely out of the fluid and with the axis AM inclined to the surface at an angle less than T .

It remains to mention the traditions regarding other investigations by Archimedes which have reached us in Greek or through the Arabic.

(α) *The Cattle-Problem.*

This is a difficult problem in indeterminate analysis. It is required to find the number of bulls and cows of each of four colours, or to find 8 unknown quantities. The first part of the problem connects the unknowns by seven simple equations ; and the second part adds two more conditions to which the unknowns must be subject. If W, w be the numbers of white bulls and cows respectively and $(X, x), (Y, y), (Z, z)$ represent the numbers of the other three colours, we have first the following equations :

$$(I) \quad W = \left(\frac{1}{2} + \frac{1}{3}\right) X + Y, \quad (\alpha)$$

$$X = \left(\frac{1}{4} + \frac{1}{5}\right) Z + Y, \quad (\beta)$$

$$Z = \left(\frac{1}{6} + \frac{1}{7}\right) W + Y, \quad (\gamma)$$

$$(II) \quad w = \left(\frac{1}{3} + \frac{1}{4}\right) (X + x), \quad (\delta)$$

$$x = \left(\frac{1}{4} + \frac{1}{5}\right) (Z + z), \quad (\epsilon)$$

$$z = \left(\frac{1}{5} + \frac{1}{6}\right) (Y + y), \quad (\zeta)$$

$$y = \left(\frac{1}{6} + \frac{1}{7}\right) (W + w). \quad (\eta)$$

Secondly, it is required that

$$W + X = \text{a square}, \quad (\theta)$$

$$Y + Z = \text{a triangular number}. \quad (i)$$

There is an ambiguity in the text which makes it just possible that $W + X$ need only be the product of two whole numbers instead of a square as in (θ) . Jul. Fr. Wurm solved the problem in the simpler form to which this change reduces it. The complete problem is discussed and partly solved by Amth

The general solution of the first seven equations is

$$W = 2 \cdot 3 \cdot 7 \cdot 53 \cdot 4657 n = 10366482 n,$$

$$X = 2 \cdot 3^2 \cdot 89 \cdot 4657 n = 7460514 n,$$

$$Y = 3^4 \cdot 11 \cdot 4657 n = 4149387 n,$$

$$Z = 2^2 \cdot 5 \cdot 79 \cdot 4657 n = 7358060 n,$$

$$w = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 23 \cdot 373 n = 7206360 n,$$

$$x = 2 \cdot 3^2 \cdot 17 \cdot 15991 n = 4893246 n,$$

$$y = 3^2 \cdot 13 \cdot 46489 n = 5439213 n,$$

$$z = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 761 n = 3515820 n.$$

It is not difficult to find such a value of n that $W + X$ is a square number; it is $n = 3 \cdot 11 \cdot 29 \cdot 4657 \xi^2 = 445674 \xi^2$ where ξ is any integer. We then have to make $Y + Z$ a triangular number, i.e. a number of the form $\frac{1}{2}q(q+1)$. This reduces itself to the solution of the 'Pellian' equation

$$t^2 - 4729494u^2 = 1,$$

which leads to prodigious figures; one of the eight unknown quantities alone would have more than 206,500 digits!

(β) *On semi-regular polyhedra.*

In addition, Archimedes investigated polyhedra of a certain type. This we learn from Pappus. The polyhedra in question

designate a polyhedron contained by m regular polygons of α sides, n regular polygons of β sides, &c., by $(m_\alpha, n_\beta \dots)$, the thirteen Archimedean polyhedra, which we will denote by $P_1, P_2 \dots P_{13}$, are as follows:

Figure with 8 faces: $P_1 \equiv (4_3, 4_6)$.

Figures with 14 faces: $P_2 \equiv (8_3, 6_4)$, $P_3 \equiv (6_4, 8_6)$,
 $P_4 \equiv (8_3, 6_8)$.

Figures with 26 faces: $P_5 \equiv (8_3, 18_4)$, $P_6 \equiv (12_4, 8_6, 6_8)$.

Figures with 32 faces: $P_7 \equiv (20_3, 12_5)$, $P_8 \equiv (12_5, 20_6)$,
 $P_9 \equiv (20_3, 12_{10})$.

Figure with 38 faces: $P_{10} \equiv (32_3, 6_4)$.

Figures with 62 faces: $P_{11} \equiv (20_3, 30_4, 12_5)$,
 $P_{12} \equiv (30_4, 20_6, 12_{10})$.

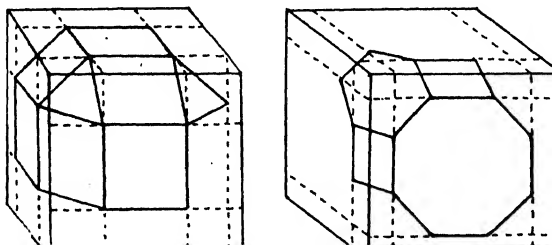
Figure with 92 faces: $P_{13} \equiv (80_3, 12_5)$.

Kepler¹ showed how these figures can be obtained. A method of obtaining some of them is indicated in a fragment of a scholium to the Vatican MS. of Pappus. If a solid angle of one of the regular solids be cut off symmetrically by a plane, i.e. in such a way that the plane cuts off the same length from each of the edges meeting at the angle, the section is a regular polygon which is a triangle, square or pentagon according as the solid angle is formed of three, four, or five plane angles. If certain equal portions be so cut off from all the solid angles respectively, they will leave regular polygons inscribed in the faces of the solid; this happens (A) when the cutting planes bisect the sides of the faces and so leave in each face a polygon of the same kind, and (B) when the cutting planes cut off a smaller portion from each angle in such a way that a regular polygon is left in each face which has double the number of sides (as when we make, say, an octagon out of a square by cutting off the necessary portions,

¹ Kepler, *Harmonice mundi in Opera* (1864), v, pp. 123-6.

symmetrically, from the corners). We have seen that in passing to Heron, two of the semi-regular solids had already been discovered by Plato, and this would doubtless be his method. The methods (A) and (B) applied to the five regular solids give the following out of the 13 semi-regular solids: (1) from the tetrahedron, P_1 by cutting off corners so as to leave hexagons in the faces; (2) from the cube, leaving squares, and P_4 by leaving octagons, in the faces; (3) from the octahedron, P_2 by leaving triangles, and P_3 by leaving hexagons, in the faces; (4) from the icosahedron, P_7 by leaving triangles, and P_8 by leaving hexagons, in the faces; (5) from the dodecahedron, P_7 by leaving pentagons, and P_9 by leaving decagons in the faces.

Of the remaining six, four are obtained by cutting off the edges symmetrically and equally by planes parallel to the faces, and then cutting off angles. Take first the cube: (1) Cut off from each of four parallel edges portions which leave an octagon as the section of the figure perpendicular to the edges; then cut off equilateral triangles from the corners (see Fig. 1); this gives P_5 containing 8 equilateral triangles and 18 squares. (P_5 is also obtained by bisecting the edges of P_2 and cutting off corners.) (2) Cut off from the edges of the cube a smaller portion so as to leave on each face a square such that the octagon described in it has a side equal to the breadth of the section in which each edge was cut; then cut off hexagons from each angle (see Fig.



produces P_{11} and P_{12} (see Figs. 3, 4 for the case of the icosahedron).

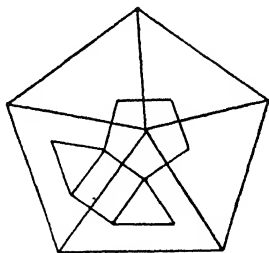


FIG. 3.

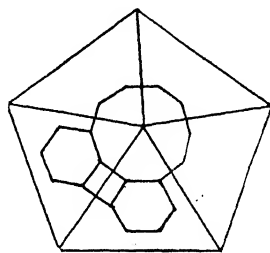


FIG. 4.

The two remaining solids P_{10} , P_{13} cannot be so simply produced. They are represented in Figs. 5, 6, which I have

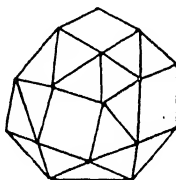


FIG. 5.

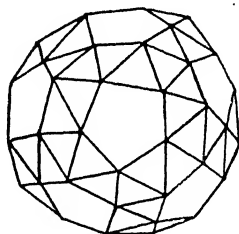


FIG. 6.

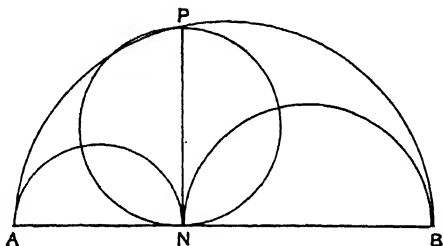
taken from Kepler. P_{10} is the *snub cube* in which each solid angle is formed by the angles of four equilateral triangles and one square; P_{13} is the *snub dodecahedron*, each solid angle of which is formed by the angles of four equilateral triangles and one regular pentagon.

We are indebted to Arabian tradition for

(γ) *The Liber Assumptorum.*

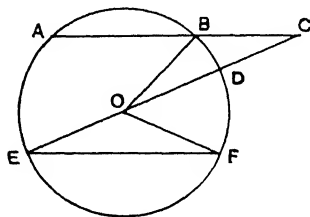
Of the theorems contained in this collection many are so elegant as to afford a presumption that they may really be due to Archimedes. In three of them the figure appears which was called $\alpha\rho\beta\eta\lambda\omicron\varsigma$, a shoemaker's knife, consisting of three semicircles with a common diameter as shown in the annexed figure. If N be the point at which the diameters

of the two smaller semicircles adjoin, and NP be right angles to AB meeting the external semicircle area of the $\alpha\rho\beta\eta\lambda\sigma$ (included between the three semicircles) is equal to the circle on PN as diameter (Prop. 5 it is shown that, if a circle be described in between the arcs AP , AN and the straight line PN



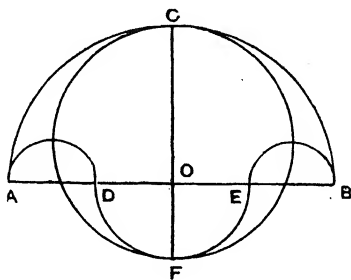
all three, and if a circle be similarly described in between the arcs PB , NB and the straight line PN all three, the two circles are equal. If one circle be in the $\alpha\rho\beta\eta\lambda\sigma$ touching all three semicircles, Prop. 6 that, if the ratio of AN to NB be given, we can find the relation between the diameter of the circle inscribed in the $\alpha\rho\beta\eta\lambda\sigma$ and the straight line AB ; the proof is for the particular case $AN = \frac{3}{2}BN$, and shows that the diameter of the inscribed circle $= \frac{6}{19}AB$.

Prop. 8 is of interest in connexion with the problem



trisecting any angle. If AB be any chord of a circle

Lastly, we may mention the elegant theorem about the area of the *Salinon* (presumably 'salt-cellar') in Prop. 14. ACB is a semicircle on AB as diameter, AD , EB are equal lengths measured from A and B on AB . Semicircles are drawn with AD , EB as diameters on the side towards C , and



a semicircle with DE as diameter is drawn on the other side of AB . CF is the perpendicular to AB through O , the centre of the semicircles ACB , DFE . Then is the area bounded by all the semicircles (the *Salinon*) equal to the circle on CF as diameter.

The Arabians, through whom the Book of Lemmas has reached us, attributed to Archimedes other works (1) on the Circle, (2) on the Heptagon in a Circle, (3) on Circles touching one another, (4) on Parallel Lines, (5) on Triangles, (6) on the properties of right-angled triangles, (7) a book of Data, (8) *De clepsydris*: statements which we are not in a position to check. But the author of a book on the finding of chords in a circle,¹ Abū'l Raihān Muḥ. al-Bīrūnī, quotes some alternative proofs as coming from the first of these works.

(8) *Formula for area of triangle.*

More important, however, is the mention in this same work of Archimedes as the discoverer of two propositions hitherto attributed to Heron, the first being the problem of finding the perpendiculars of a triangle when the sides are given, and

appropriate place for ERATOSTHENES of Cyrene. It is that Archimedes dedicated *The Method*, and the *Catoptrics* purports, by its heading, to have been sent through the mathematicians of Alexandria. It is evident from the preface to *The Method* that Archimedes thought highly of his mathematical ability. He was, indeed, recognized by his contemporaries as a man of great distinction in all branches of knowledge, though in each subject he just fell short of the highest place. On the latter ground he was called by another nickname applied to him, *Pentathlos*, having the implication, representing as it does an all-round athlete, that he was not the first runner or wrestler but took the second place in these contests as well as in others. He was younger than Archimedes; the date of his birth was about 284 B.C. or thereabouts. He was a pupil of the philosopher Ariston of Chios, the grammarian Lysanias of Cyrene, the poet Callimachus; he is said also to have been a student of Zeno the Stoic, and he may have come under the influence of Arcesilaus at Athens, where he spent a considerable part of his life. Invited, when about 40 years of age, by Ptolemy III. to be tutor to his son (Philopator), he became librarian of Alexandria; his obligation to Ptolemy he recognized by a column which he erected with a graceful epigram inscribed on it. This is the epigram, with which we are already acquainted (vol. i, p. 260), relating to the solutions, discovered by him of the problem of the duplication of the cube, and in which he is giving his own method by means of an appliance called *loxos*, itself represented in bronze on the column.

Eratosthenes wrote a book with the title *Πλατωνικά*, whether it was a sort of commentary on the *Timæus* of Plato, or a dialogue in which the principal part was played by Plato, it evidently dealt with the fundamental principles of mathematics in connexion with Plato's philosophy, and was naturally one of the important sources of Theon of Smyrna's work on the mathematical matters which it was necessary for the student of Plato to know; and Theon cites it several times by name. It seems to have begun with the solution of the problem of Delos, telling the story quoted by Theon, that the god required, as a means of stopping a plague, that

there, which was cubical in form, should be doubled in size. The book evidently contained a disquisition on *proportion* (*ἀναλογία*); a quotation by Theon on this subject shows that Eratosthenes incidentally dealt with the fundamental definitions of geometry and arithmetic. The principles of music were discussed in the same work.

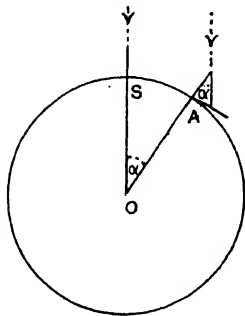
We have already described Eratosthenes's solution of the problem of Delos, and his contribution to the theory of arithmetic by means of his *sieve* (*κόσκινον*) for finding successive prime numbers.

He wrote also an independent work *On means*. This was in two Books, and was important enough to be mentioned by Pappus along with works by Euclid, Aristaeus and Apollonius as forming part of the *Treasury of Analysis*¹; this proves that it was a systematic geometrical treatise. Another passage of Pappus speaks of certain loci which Eratosthenes called 'loci with reference to means' (*τόποι πρὸς μεσότητας*)²; these were presumably discussed in the treatise in question. What kind of loci these were is quite uncertain; Pappus (if it is not an interpolator who speaks) merely says that these loci 'belong to the aforesaid classes of loci', but as the classes are numerous (including 'plane', 'solid', 'linear', 'loci on surfaces', &c.), we are none the wiser. Tannery conjectured that they were loci of points such that their distances from three fixed straight lines furnished a 'médieté', i.e. loci (straight lines and conics) which we should represent in trilinear coordinates by such equations as $2y = x + z$, $y^2 = xz$, $y(x + z) = 2xz$, $x(x - y) = z(y - z)$, $x(x - y) = y(y - z)$, the first three equations representing the arithmetic, geometric and harmonic means, while the last two represent the 'subcontraries' to the harmonic and geometric means respectively. Zeuthen has a different conjecture.³ He points out that, if QQ' be the polar of a given point C with reference to a conic, and $CPOP'$ be drawn through C meeting QQ' in O and the conic in P, P' , then CO is the harmonic mean to CP, CP' ; the locus of O for all transversals CPP' is then the straight line QQ' . If A, G

geometric mean between CP , CP' , the loci of A , G respectively are conics. Zeuthen therefore suggests that these loci be the corresponding loci of the points on CPP' at a distance from C equal to the subcontraries of the geometric harmonic means between CP and CP' are the 'loci of reference to means' of Eratosthenes; the latter two loci are 'linear', i.e. higher curves than conics. Needless to say, we have no confirmation of this conjecture.

Eratosthenes's measurement of the Earth.

But the most famous scientific achievement of Eratosthenes was his measurement of the earth. Archimedes mentions that we have seen, that some had tried to prove that the circumference of the earth is about 300,000 stades. This was evidently the measurement based on observations made at Lysimachia (on the Hellespont) and Syene. It was observed that, while both these places were on one meridian, the star of Draco was in the zenith at Lysimachia, and Cancer in the zenith at Syene; the arc of the meridian separating the two places in the heavens was taken to be $1/15$ th of the complete circle.



The distance between the two places was estimated at 20,000 stades, and accordingly the whole circumference of the earth was reckoned at 300,000 stades. Eratosthenes improved on this. He observed (1) that at Syene at noon, at the summer solstice, the sun cast no shadow from an upright gnomon (this was confirmed by observation that a well dug at the same place was entirely lighted up at the same time), while (2) at the same moment the gnomon set upright at Alexandria (taken to be on the same meridian as Syene) cast a shadow corresponding to an angle between

α , or $1/50$ th of four right angles. Now the distance from S to A was known by measurement to be 5,000 stades; it followed that the circumference of the earth was 250,000 stades. This is the figure given by Cleomedes, but Theon of Smyrna and Strabo both give it as 252,000 stades. The reason of the discrepancy is not known; it is possible that Eratosthenes corrected 250,000 to 252,000 for some reason, perhaps in order to get a figure divisible by 60 and, incidentally, a round number (700) of stades for one degree. If Pliny is right in saying that Eratosthenes made 40 stades equal to the Egyptian $\sigma\chi\omicron\iota\nu\omicron\varsigma$, then, taking the $\sigma\chi\omicron\iota\nu\omicron\varsigma$ at 12,000 Royal cubits of 0.525 metres, we get 300 such cubits, or 157.5 metres, i.e. 516.73 feet, as the length of the stade. On this basis 252,000 stades works out to 24,662 miles, and the diameter of the earth to about 7,850 miles, only 50 miles shorter than the true polar diameter, a surprisingly close approximation, however much it owes to happy accidents in the calculation.

We learn from Heron's *Dioptra* that the measurement of the earth by Eratosthenes was given in a separate work *On the Measurement of the Earth*. According to Galen¹ this work dealt generally with astronomical or mathematical geography, treating of 'the size of the equator, the distance of the tropic and polar circles, the extent of the polar zone, the size and distance of the sun and moon, total and partial eclipses of these heavenly bodies, changes in the length of the day according to the different latitudes and seasons'. Several details are preserved elsewhere of results obtained by Eratosthenes, which were doubtless contained in this work. He is supposed to have estimated the distance between the tropic circles or twice the obliquity of the ecliptic at $11/83$ rd of a complete circle or $47^{\circ} 42' 39''$; but from Ptolemy's language on this subject it is not clear that this estimate was not Ptolemy's own. What Ptolemy says is that he himself found the distance between the tropic circles to lie always between $47^{\circ} 40'$ and $47^{\circ} 45'$, 'from which we obtain *about*

mean of Ptolemy's estimates, $47\frac{1}{2} - 50$, is of course $11\frac{1}{2}/83$ ths of 360° . It is consistent with Ptolemy's lang to suppose that Eratosthenes adhered to the value of obliquity of the ecliptic discovered before Euclid's, namely 24° , and Hipparchus does, in his extant *Commentary on the Phaenomena of Aratus and Eudoxus*, say that summer tropic is 'very nearly 24° north of the equator'.

The *Doxographi* state that Eratosthenes estimated distance of the moon from the earth at 780,000 stades the distance of the sun from the earth at 804,000,000 stades (the versions of Stobaeus and Joannes Lydus admit 4,080,000 as an alternative for the latter figure, but this obviously cannot be right). Macrobius² says that Eratosthenes made the 'measure' of the sun to be 27 times that of the earth. It is not certain whether measure means 'solid content' or 'diameter' in this case; the other figures on record make the former more probable, in which case the diameter of the sun would be three times that of the earth. Macrobius also tells us that Eratosthenes's estimates of the distances of the sun and moon were obtained by means of lunar eclipses.

Another observation by Eratosthenes, namely that at Syene (which is under the summer tropic) and throughout a circle round it with a radius of 300 stades the upright gnomon throws no shadow at noon, was afterwards made use of by Posidonius in his calculation of the size of the sun. Assuming that the circle in which the sun apparently moves round the earth is 10,000 times the size of a circular section of the earth through its centre, and combining with this hypothesis the datum just mentioned, Posidonius arrived at 3,000,000 stades as the diameter of the sun.

Eratosthenes wrote a poem called *Hermes* containing a good deal of descriptive astronomy; only fragments of this have survived. The work *Catasterismi* (literally 'placings and names of the stars') which is extant can hardly be genuine in the form in which it has reached us; it goes back, however, to a genuine work by Eratosthenes which apparently bore the same name. Alternatively it is alluded to as *Κατάλογοι* or by the gen.

¹ Ptolemy, *Syntaxis*, i. 12, pp. 67. 22-68. 6.

² Macrobius, *In Somn. Scip.* i. 20. 9.

word *Ἀστρονομία* (Suidas), which latter word is perhaps a mistake for *Ἀστροθεσία* corresponding to the title *Ἀστροθεσίαι ζῳδίων* found in the manuscripts. The work as we have it contains the story, mythological and descriptive, of the constellations, &c., under forty-four heads; there is little or nothing belonging to astronomy proper.

Eratosthenes is also famous as the first to attempt a scientific chronology beginning from the siege of Troy; this was the subject of his *Χρονογραφίαι*, with which must be connected the separate *Ὀλυμπιονίκαι* in several books. Clement of Alexandria gives a short *résumé* of the main results of the former work, and both works were largely used by Apollodorus. Another lost work was on the Octaëteris (or eight-years' period), which is twice mentioned, by Geminus and Achilles; from the latter we learn that Eratosthenes regarded the work on the same subject attributed to Eudoxus as not genuine. His *Geographica* in three books is mainly known to us through Suidas's criticism of it. It began with a history of geography down to his own time; Eratosthenes then proceeded to mathematical geography, the spherical form of the earth, the negligibility in comparison with this of the unevennesses caused by mountains and valleys, the changes of features due to floods, earthquakes and the like. It would appear from Theon of Smyrna's allusions that Eratosthenes estimated the height of the highest mountain to be 10 stades or about $1/8000$ th part of the diameter of the earth.

XIV

CONIC SECTIONS. APOLLONIUS OF PERGAE

A. HISTORY OF CONICS UP TO APOLLONIUS

Discovery of the conic sections by Menaechmus

WE have seen that Menaechmus solved the problem of finding two mean proportionals (and therefore the duplication of the cube) by means of conic sections, and that he is credited with the discovery of the three curves; for the epigram of Eratosthenes speaks of 'the *triads* of Menaechmus', where, of course only two conics, the parabola and the rectangular hyperbola, actually appear in Menaechmus's solutions. The question arises, how did Menaechmus come to think of finding curves by cutting a cone? On this we have no information whatever. Democritus had indeed spoken of a section of a cone parallel and very near to the base, which of course would be a circle, since the cone would certainly be then a circular cone. But it is probable enough that the attention of the Greeks, whose observation nothing escaped, was attracted to the shape of a section of a cone or a cylinder by a plane obliquely inclined to the axis when it occurred often would, in real life; the case where the solid was cut right through, which would show an ellipse, would probably be noticed first, and some attempt would be made to investigate the nature and geometrical measure of the section of the figure in relation to the circular sections of the same solid; these would in the first instances be made

as well as a cone is actually made by Euclid in his *Phaenomena*: 'if', says Euclid, 'a cone or a cylinder be cut by a plane not parallel to the base, the resulting section is a section of an acute-angled cone which is similar to a *θυρεός* (shield).' After this would doubtless follow the question what sort of curves they are which are produced if we cut a cone by a plane which does not cut through the cone completely, but is either parallel or not parallel to a generator of the cone, whether these curves have the same property with the ellipse and with one another, and, if not, what exactly are their fundamental properties respectively.

As it is, however, we are only told how the first writers on conics obtained them in actual practice. We learn on the authority of Geminus¹ that the ancients defined a cone as the surface described by the revolution of a right-angled triangle about one of the sides containing the right angle, and that they knew no cones other than right cones. Of these they distinguished three kinds; according as the vertical angle of the cone was less than, equal to, or greater than a right angle, they called the cone acute-angled, right-angled, or obtuse-angled, and from each of these kinds of cone they produced one and only one of the three sections, the section being always made perpendicular to one of the generating lines of the cone; the curves were, on this basis, called 'section of an acute-angled cone' (= an ellipse), 'section of a right-angled cone' (= a parabola), and 'section of an obtuse-angled cone' (= a hyperbola) respectively. These names were still used by Euclid and Archimedes.

Menaechmus's probable procedure.

Menaechmus's constructions for his curves would presumably be the simplest and the most direct that would show the desired properties, and for the parabola nothing could be simpler than a section of a right-angled cone by a plane at right angles to one of its generators. Let *OBC* (Fig. 1) represent

¹ Eutocius, *Comm. on Conics* of Apollonius.

conceive a section through AO (perpendicular to the plane of the paper) at right angles to the plane of the paper.

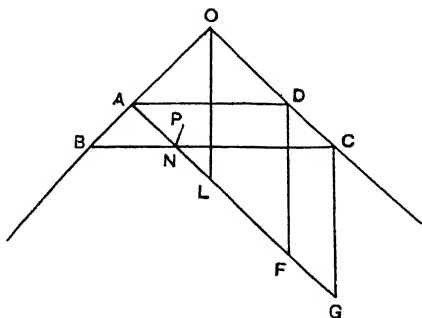


FIG. 1.

If P is any point on the curve, and PN perpendicular to BC , let BC be drawn through N perpendicular to the plane of the cone. Then P is on the circular section of the cone with BC as diameter.

Draw AD parallel to BC , and DF, CG parallel to AL produced in F, G . Then AD, AF are both perpendicular to OL .

$$\begin{aligned} \text{If now} \quad & PN = y, \quad AN = x, \\ & y^2 = PN^2 = BN \cdot NC. \end{aligned}$$

But B, A, C, G are concyclic, so that

$$\begin{aligned} BN \cdot NC &= AN \cdot NG \\ &= AN \cdot AF \\ &= AN \cdot 2AL. \end{aligned}$$

$$\begin{aligned} \text{Therefore} \quad & y^2 = AN \cdot 2AL \\ & = 2AL \cdot x, \end{aligned}$$

and $2AL$ is the 'parameter' of the principal ordinate.
In the case of the hyperbola Menaechmus had to

But, by similar triangles,

$$\begin{aligned} NG:AF &= NC:AD \\ &= A'N:AA'. \end{aligned}$$

$$\begin{aligned} \text{Hence} \quad PN^2 &= AN \cdot A'N \cdot \frac{AF}{AA'} \\ &= \frac{2AL}{AA'} \cdot AN \cdot A'N, \end{aligned}$$

which is the property of the hyperbola, AA' being call the transverse axis, and $2AL$ the parameter of the ordinates.

Now, in order that the hyperbola may be rectangular must have $2AL:AA'$ equal to 1. The problem then is: given a straight line AA' , and AL along $A'A$ equal to $\frac{1}{2}AA'$, to find a cone such that L is on its section through AL perpendicular to the generator. A is a rectangular hyperbola with $A'A$ as transverse axis. In other words, we have to find a point O on the straight line through A perpendicular to AA' such that OL be an angle which is the supplement of the angle $A'O A$.

This is the case if $A'O:OA = A'L:LA = 3:1$; therefore O is on the circle which is the locus of points such that their distances from the two fixed points A and A' are in the ratio 3:1. This circle is the circle with AA' as diameter, where $A'K:KA = A'L:LA = 3:1$. L is on this circle, and O is then determined as the point in which the line drawn perpendicular to AA' intersects the circle.

It is to be observed, however, that this deduction of the particular from a more general case is not usual in Greek mathematics; on the contrary, the particular is usually led to the more general. Notwithstanding, therefore, the orthodox method of producing conic sections is satisfied by cutting the generator of each cone perpendicularly. I am inclined to think that Menaechmus would generate hyperbolas directly and in an easier way.

For, let the right-angled cone HOK (Fig. 3) be cut by a plane through $A'A$ parallel to the axis OM and cutting the sides of the axial triangle HOK in A' , A , N respectively. Let P be the point on the curve for which PN is the principal ordinate. Draw OC parallel to HK . We have at once

$$\begin{aligned} PN^2 &= HN \cdot NK \\ &= MK^2 - MN^2 \end{aligned}$$

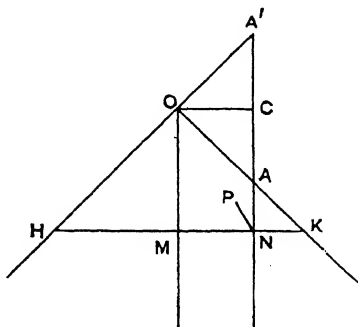
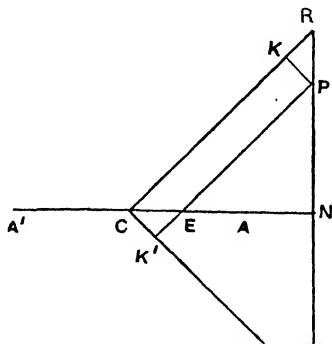


FIG. 3.

$$= CN^2 - CA^2, \text{ since } MK = OM, \text{ and } MN = OC = CA.$$

This is the property of the rectangular hyperbola having $A'A$ as axis. To obtain a particular rectangular hyperbola with axis of given length we have only to choose the cutting plane so that the intercept $A'A$ may have the given length.

But Menaechmus had to prove the asymptote-property of his rectangular hyperbola. As he can hardly be supposed to have got as far as Apollonius in investigating the relations of the hyperbola to its asymptotes, it is probably safe to assume that he obtained the particular property in the simplest way, i. e. directly from the property of the curve in relation to its axes.



at right angles) and $A'A$ the axis of a rectangular hyperbola. Let P any point on the curve, PN the principal ordinate, PK, PK' perpendicular to the asymptotes respectively, PN produced meet the asymptotes in R, R' .

Now, by the axial property,

$$CA^2 = CN^2 - PN^2$$

$$= RN^2 - PN^2$$

$$= RP \cdot PR'$$

$$= 2PK \cdot PK', \text{ since } \angle PRK \text{ is half a right angle}$$

therefore $PK \cdot PK' = \frac{1}{2}CA^2$.

Works by Aristaeus and Euclid.

If Menaechmus was really the discoverer of the three conic sections at a date which we must put at about 360 or 350 B.C., the subject must have been developed very rapidly, for by the end of the century there were two considerable works on conics in existence, works which, as we learn from Pappus, were considered worthy of a place, alongside the works of Apollonius, in the *Treasury of Analysis*. Euclid's *Conics* in four books was preceded by a work of Aristaeus, which was still extant in the time of Pappus, who describes it as 'five books of *Solid Loci* connected (or continuous) with the conics'. Speaking of the relation of Euclid's *Conics* in four books to this work, Pappus says (if the work is genuine) that Euclid gave credit to Aristaeus for his discoveries in conics and did not attempt to anticipate him, but wished to construct anew the same system. In preface to Euclid, when dealing with what Apollonius calls the three- and four-line locus, 'wrote so much about the locus as was possible by means of the conics of Aristaeus; without completeness for his demonstrations'.¹ We gather from these remarks that Euclid's *Conics* was a compilation and refinement of the geometry of the conics so far as known

¹ Pappus, vii. p. 678. 4.

whereas the work of Aristaeus was more specialized and original.

'Solid loci' and 'solid problems'.

'Solid loci' are of course simply conics, but the use of the 'Solid loci' instead of 'conics' seems to indicate that the work was in the main devoted to conics regarded as loci. As we have seen, 'solid loci' which are conics are distinguished from 'plane loci', on the one hand, which are straight lines and circles, and from 'linear loci' on the other, which are of a higher order than conics. There is some doubt as to the reason why the term 'solid loci' was applied to the conics. We are told that 'plane' loci are so called because they are generated in a plane (but so are some of the higher curves, such as the *quadratrix* and the spiral of Archimedes), and that 'solid loci' derived their name from the fact that they arise as sections of solid figures (but so do some higher curves, e.g. the spiric curves which are sections of the *σπεῖρα* or *torus*). But some light is thrown on the subject by the corresponding distinction which Pappus draws between 'plane', 'solid' and 'linear' *problems*.

In discussing 'solid problems', he says, 'which can be solved by means of a straight line and a circumference of a circle may properly be called *plane*; for the lines by means of which such problems are solved have their origin in a plane. Those, however, which are solved by using for their discovery one or more of the sections of the cone have been called *solid*; for their construction requires the use of surfaces of solid figures, and not only those of cones. There remains a third kind of problem, that which is called *linear*; for other lines (curves) besides those mentioned are assumed for the construction, the nature of which is more complicated and less natural, as they are generated from more irregular surfaces and intricate constructions.'¹

The true significance of the word 'plane' as applied to

areas, manipulation of simple equations between areas; in particular, the application of areas; in other words, problems were those which, if expressed algebraically, were on equations of a degree not higher than the second. Problems, however, soon arose which did not yield to these methods. One of the first was that of the duplication of the cube, which was a problem of geometry in three dimensions, solid geometry. Consequently, when it was found that the problem could be solved by means of conics, and that higher curves were necessary, it would be natural to call them as 'solid' loci, especially as they were in fact derived from sections of a solid figure, the cone. The proper term would be only confirmed when it was found that the duplication of the cube depended on the solution of a cubic equation, other problems such as the trisection of an angle, or the cutting of a sphere into two segments in a given ratio to one another, led to an equation of the third degree in volumes in one form or another, i.e. a mixed cubic equation, and that this equation, which was also a solid problem, could likewise be solved by means of conics.

Aristaeus's *Solid Loci*.

The *Solid Loci* of Aristaeus, then, presumably included problems whose loci proved to be conic sections. In particular, we have discussed, however imperfectly, the locus with respect to three or four lines the synthesis of which Apollonius had found inadequately worked out in Euclid's *Conics*. The theorems relating to this locus are enunciated by Aristaeus in this way:

'If three straight lines be given in position and from a fixed point straight lines be drawn to meet the three straight lines at given angles, and if the ratio of the rectangle contained by two of the straight lines so drawn to the square on the remaining one be given, then the point will

same way the point will lie on a conic section given in
ion.'¹

The reason why Apollonius referred in this connexion to
id and not to Aristæus was probably that it was Euclid's
k that was on the same lines as his own.

A very large proportion of the standard properties of conics
t of being stated in the form of locus-theorems; if a
in property holds with regard to a certain point, then
point lies on a conic section. But it may be assumed
Aristæus's work was not merely a collection of the
ary propositions transformed in this way; it would deal
new locus-theorems not implied in the fundamental
itions and properties of the conics, such as those just
ioned, the theorems of the three- and four-line locus.
one (to us) ordinary property, the focus-directrix property,
as it seems to me, in all probability included.

Focus-directrix property known to Euclid.

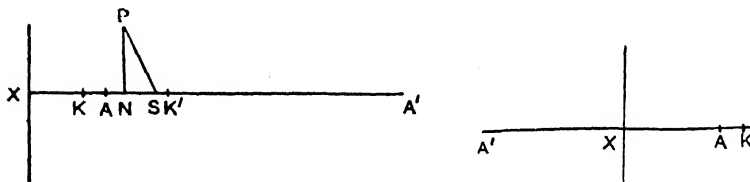
It is remarkable that the directrix does not appear at all in
lonius's great treatise on conics. The focal properties of
central conics are given by Apollonius, but the foci are
ned in a different way, without any reference to the
trix; the focus of the parabola does not appear at all.
may perhaps conclude that neither did Euclid's *Conics*
in the focus-directrix property; for, according to Pappus,
lonius based his first four books on Euclid's four books,
e filling them out and adding to them. Yet Pappus gives
proposition as a lemma to Euclid's *Surface-Loci*, from
h we cannot but infer that it was assumed in that
ise without proof. If, then, Euclid did not take it from
wn *Conics*, what more likely than that it was contained
ristæus's *Solid Loci*?

Pappus's enunciation of the theorem is to the effect that the
of a point such that its distance from a given point is in

Proof from Pappus.

The proof in the case where the given ratio is different unity is shortly as follows.

Let S be the fixed point, SX the perpendicular from the fixed line. Let P be any point on the locus and



perpendicular to SX , so that SP is to NX in the ratio (e);

thus
$$e^2 = (PN^2 + SN^2) : NX^2.$$

Take K on SX such that

$$e^2 = SN^2 : NK^2;$$

then, if K' be another point on SN , produced if necessary such that $NK = NK'$,

$$\begin{aligned} e^2 : 1 &= (PN^2 + SN^2) : NX^2 = SN^2 : NK^2 \\ &= PN^2 : (NX^2 - NK^2) \\ &= PN^2 : XK \cdot XK'. \end{aligned}$$

The positions of N , K , K' change with the position of P . If A , A' be the points on which N falls when K , K' coincide with X respectively, we have

$$SA : AX = SN : NK = e : 1 = SN : NK' = SA' : A'X$$

Similarly it can be shown that

$$(1 \sim e):e = XK':A'N.$$

By multiplication, $XK \cdot XK':AN \cdot A'N = (1 \sim e^2):e^2$;

and it follows from above, *ex aequali*, that

$$PN^2:AN \cdot A'N = (1 \sim e^2):1,$$

which is the property of a central conic.

When $e < 1$, A and A' lie on the same side of X , while N lies on AA' , and the conic is an ellipse; when $e > 1$, A and A' lie on opposite sides of X , while N lies on $A'A$ produced, and the conic is a hyperbola.

The case where $e = 1$ and the curve is a parabola is easy and need not be reproduced here.

The treatise would doubtless contain other loci of types similar to that which, as Pappus says, was used for the trisection of an angle: I refer to the proposition already quoted (vol. i, p. 243) that, if A , B are the base angles of a triangle with vertex P , and $\angle B = 2\angle A$, the locus of P is a hyperbola with eccentricity 2.

Propositions included in Euclid's *Conics*.

That Euclid's *Conics* covered much of the same ground as the first three Books of Apollonius is clear from the language of Apollonius himself. Confirmation is forthcoming in the quotations by Archimedes of propositions (1) 'proved in the elements of conics', or (2) assumed without remark as already known. The former class include the fundamental ordinate properties of the conics in the following forms:

(1) for the ellipse,

$$PN^2:AN \cdot A'N = P'N'^2:AN' \cdot A'N' = BC^2:AC^2;$$

(2) for the hyperbola,

$$PN^2:AN \cdot A'N = P'N'^2:AN' \cdot A'N';$$

angles contained by the segments of the chords respectively are to one another as the squares of the parallel tangents to the parabola. This is not the by no means easy proposition that, if in a parabola a diameter through P bisects the chord QQ' in V , and QV drawn perpendicular to PV , then

$$QV^2:QD^2 = p:p_a,$$

where p_a is the parameter of the principal ordinates and p the parameter of the ordinates to the diameter PV .

Conic sections in Archimedes.

But we must equally regard Euclid's *Conics* as the source from which Archimedes took most of the other ordinary properties of conics which he assumes without proof. Even in summarizing these it will be convenient to refer to Archimedes's terminology. We have seen that the axes of an ellipse are not called axes but *diameters*, greater and less; the axis of a parabola is likewise its *diameter* and the lines parallel to it are 'diameters', although in a segment of a parabola the diameter bisecting the base is the 'diameter' of the segment. The two 'diameters' of an ellipse are *conjugate*. In the case of the hyperbola the 'diameter' (axis) is the portion of it within the (single-branched) hyperbola; the centre is not called the 'centre', but the point in which the 'nearest lines to the section of an obtuse-angled cone' (the asymptotes) meet; the half of the axis (CA) is 'the line adjacent to the axis' (of the hyperboloid of revolution obtained by making the hyperbola revolve about its 'diameter' and $A'A$ is double of this line. Similarly CP is the 'adjacent to the axis' of a segment of the hyperboloid and $P'P$ double of this line. It is clear that Archimedes did not yet treat the two branches of a hyperbola as forming a single curve; this was reserved for Apollonius.

The main properties of conics assumed by Archimedes in addition to those above mentioned may be summarized thus:

Central Conics.

1. The property of the ordinates to any diameter PP' ,

$$QV^2:PV.P'V = Q'V'^2:PV'.P'V'.$$

In the case of the hyperbola Archimedes does not give any expression for the constant ratios $PN^2:AN.A'N$ and $QV^2:PV.P'V$ respectively, whence we conclude that he had no conception of diameters or radii of a hyperbola not meeting the curve.

2. The straight line drawn from the centre of an ellipse, or the point of intersection of the asymptotes of a hyperbola, through the point of contact of any tangent, bisects all chords parallel to the tangent.
3. In the ellipse the tangents at the extremities of either of two conjugate diameters are both parallel to the other diameter.
4. If in a hyperbola the tangent at P meets the transverse axis in T , and PN is the principal ordinate, $AN > AT$. (It is not easy to see how this could be proved except by means of the general property that, if PP' be any diameter of a hyperbola, QV the ordinate to it from Q , and QT the tangent at Q meeting $P'P$ in T , then $TP:TP' = PV:P'V$.)
5. If a cone, right or oblique, be cut by a plane meeting all the generators, the section is either a circle or an ellipse.
6. If a line between the asymptotes meets a hyperbola and is bisected at the point of concurrence, it will touch the hyperbola.
7. If x, y are straight lines drawn, in fixed directions respectively, from a point on a hyperbola to meet the asymptotes, the rectangle xy is constant.
8. If PN be the principal ordinate of P , a point on an ellipse, and if NP be produced to meet the auxiliary circle in p , the ratio $pN:PN$ is constant.
9. The criteria of similarity of conics and segments of conics are assumed in practically the same form as Apollonius gives them.

The Parabola.

1. The fundamental properties appear in the alternative forms

2. Parallel chords are bisected by one straight line parallel to the axis, which passes through the point of contact of a tangent parallel to the chords.

3. If the tangent at Q meet the diameter PV in T , and the ordinate to the diameter, $PV = PT$.

By the aid of this proposition a tangent to the parabola can be drawn (a) at a point on it, (b) parallel to a given chord.

4. Another proposition assumed is equivalent to the proposition that the square of the ordinate is equal to the rectangle of the abscissa and the subnormal, $NG = \frac{1}{2}p_a$.

5. If QQ' be a chord of a parabola perpendicular to the axis and meeting the axis in M , while QVq another chord perpendicular to the tangent at P meets the diameter through P in V , and RHK is the principal ordinate of any point R on the parabola meeting PV in H and the axis in K , then $PV:PM = MK:KA$; 'for this is proved' (*On Floating Bodies*, Book II).

Where it was proved we do not know; the proof is altogether easy.¹

6. All parabolas are similar.

As we have seen, Archimedes had to specialize the properties of the parabola for the purpose of his treatises on the *Quadrature of the Parabola*, *Conoids and Spheroids*, *Floating Bodies*, Book II, and *Plane Equilibriums*, Book II; consequently he had to prove for himself a number of special propositions which have already been given in their proper places. A few of these are assumed without proof, doubtless as being easily deducible from the propositions which he does prove. They refer to similar parabolic segments so placed that their bases are on one straight line and have one common extremity.

1. If any three similar and similarly situated parabolic segments BQ_1 , BQ_2 , BQ_3 lying along the same straight line as bases ($BQ_1 < BQ_2 < BQ_3$), and if E be any point on the tangent at B to one of the segments, and EO a straight line through E parallel to the axis of one of the segments meeting the segments in R_3 , R_2 , R_1 respectively and in O , then

$$R_3R_2:R_2R_1 = (Q_2Q_3:BQ_3) \cdot (BQ_1:Q_1Q_2).$$

¹ See *Apollonius of Perga*, ed. Heath, p. liv.

two similar parabolic segments with bases BQ_1, BQ_2 be
 as in the last proposition, and if BR_1R_2 be any straight
 through B meeting the segments in R_1, R_2 respectively,

$$BQ_1 : BQ_2 = BR_1 : BR_2.$$

These propositions are easily deduced from the theorem
 found in the *Quadrature of the Parabola*, that, if through E ,
 point on the tangent at B , a straight line ERO be drawn
 parallel to the axis and meeting the curve in R and any chord
 through B in O , then

$$ER : RO = BO : OQ.$$

To the strength of these propositions Archimedes assumes
 solution of the problem of placing, between two parabolic
 segments similar to one another and placed as in the above
 positions, a straight line of a given length and in a direction
 parallel to the diameters of either parabola.

Euclid and Archimedes no doubt adhered to the old method
 regarding the three conics as arising from sections of three
 of right circular cones (right-angled, obtuse-angled and
 acute-angled) by planes drawn in each case at right angles to
 the axis of the cone. Yet neither Euclid nor Archimedes
 were unaware that the 'section of an acute-angled cone', or
 ellipse, could be otherwise produced. Euclid actually says in
 the *Phaenomena* that 'if a cone or cylinder (presumably right)
 be cut by a plane not parallel to the base, the resulting section
 is an ellipse'. Archimedes knew that the non-circular
 sections even of an oblique circular cone made by planes
 perpendicular to all the generators are ellipses; for he shows us how,
 to draw an ellipse, to draw a cone (in general oblique) of which
 the section is an ellipse and which has its vertex outside the plane
 of the ellipse on any straight line through the centre of the
 ellipse in a plane at right angles to the ellipse and passing
 through one of its axes, whether the straight line is itself
 perpendicular or not perpendicular to the plane of the ellipse.

for the other two conics, the hyperbola and parabola, and we can scarcely avoid the inference that Archimedes was equally aware that the parabola and the hyperbola could be found otherwise than by the old method.

The first, however, to base the theory of conics on the production of all three in the most general way from a kind of circular cone, right or oblique, was Apollonius, whose work we now come.

B. APOLLONIUS OF PERGA

Hardly anything is known of the life of Apollonius except that he was born at Perga, in Pamphylia, that he went when quite young to Alexandria, where he studied with the successors of Euclid and remained a long time, and that he flourished (*γέγονε*) in the reign of Ptolemy Euergetes (247–222 B.C.). Ptolemaeus Chennus mentions an astronomer of the same name, who was famous during the reign of Ptolemy Philopator (222–205 B.C.), and it is clear that of Apollonius is meant. As Apollonius dedicated the fourth and following Books of his *Conics* to King Attalus I (241–197 B.C.), we have a confirmation of his approximate date. He was probably born about 262 B.C., or 25 years after Archimedes. We hear of a visit to Pergamum, where he made the acquaintance of Eudemus of Pergamum, to whom he dedicated the first two Books of the *Conics* in the form in which they have come down to us; they were the first two instalments of the second edition of the work.

The text of the *Conics*.

The *Conics* of Apollonius was at once recognized as an authoritative treatise on the subject, and later writers regularly cited it when quoting propositions in conics. Pappus wrote a number of lemmas to it; Serenus wrote a commentary, as also, according to Suidas, did Hypatia. Eutocius (fl. A.D. 500) prepared an edition of the first four Books and wrote a commentary on them; it is evident that he had before

the edition of Eutocius suffered interpolations which were probably made in the ninth century when, under the auspices of the emperor, mathematical studies were revived at Constantinople; it was at that date that the uncial manuscripts were copied, from which our best manuscripts, V (= Cod. Vat. gr. 204 of the twelfth to thirteenth century) for the *Conics*, and Cod. Vat. gr. 204 of the tenth century) for Eutocius, were copied.

Only the first four Books survive in Greek; the eighth is altogether lost, but the three Books V–VII exist in Arabic. It was Aḥmad and al-Ḥasan, two sons of Muḥ. b. al-Ḥasan b. Shākir, who first contemplated translating the *Conics* into Arabic. They were at first deterred by the bad state of the Greek manuscripts; but afterwards Aḥmad obtained in Syria a copy of Eutocius's edition of Books I–IV and had them translated by Hilāl b. Abī Hilāl al-Ḥimṣī (died 883/4). Books V–VII were translated, also for Aḥmad, by Thābit al-Baṣrī (826–901) from another manuscript. Naṣīraddīn's edition of this translation of the seven Books, made in 1248, was presented by two copies in the Bodleian, one of the year 1626 (No. 943) and the other of 1626 containing Books V–VII (No. 885).

A Latin translation of Books I–IV was published by Johannes Baptista Memus at Venice in 1537; but the first important edition was the translation by Commandinus (Padua, 1566), which included the lemmas of Pappus and a commentary of Eutocius, and was the first attempt to make the book intelligible by means of explanatory notes. In the Greek text Commandinus used Cod. Marcianus 518 (perhaps also Vat. gr. 205, both of which were copies of V, not V itself).

The first published version of Books V–VII was a Latin translation by Abraham Echellensis and Giacomo Alfonso Gherardi (Florence, 1661) of a reproduction of the Books written in Arabic by Abū 'l Fath al-Isfahānī.

The *editio princeps* of the Greek text is the monumental

Latin. Gregory, however, died while the work was proceeding and Halley then undertook responsibility for the whole. The Greek manuscripts used were two, one belonging to Savile and the other lent by D. Baynard; their whereabouts cannot apparently now be traced, but they were both copies of P. gr. 2356, which was copied in the sixteenth century from P. gr. 2357 of the sixteenth century, itself a copy of V. For the three Books in Arabic Halley used the Bodleian MS. 885, but also consulted (a) a compendium of the three Books by 'Abd al-melik al-Shirāzī (twelfth century), also in the Bodleian (911) (b) Borelli's edition, and (c) Bodl. 943 above mentioned, by means of which he revised and corrected his translation when completed. Halley's edition is still, so far as I know, the only available source for Books V–VII, except for the beginning of Book V (up to Prop. 7) which was edited by L. Nix (Leipzig 1889).

The Greek text of Books I–IV is now available, with the commentaries of Eutocius, the fragments of Apollonius, &c. in the definitive edition of Heiberg (Teubner, 1891–3).

Apollonius's own account of the *Conics*.

A general account of the contents of the great work which, according to Geminus, earned for him the title of the 'great geometer' cannot be better given than in the words of the writer himself. The prefaces to the several Books contain interesting historical details, and, like the prefaces of Archimedes, state quite plainly and simply in what way the treatise differs from those of his predecessors, and how much in it is claimed as original. The strictures of Pappus (or more probably his interpolator), who accuses him of being a braggart and unfair towards his predecessors, are evidently unfounded. The prefaces are quoted by v. Wilamowitz-Moellendorff as specimens of admirable Greek, showing how perfect the style of the great mathematicians could be when they were free from the trammels of mathematical terminology.

During the time I spent with you at Pergamum I observed your eagerness to become acquainted with my work in conics; I am therefore sending you the first book, which I have corrected, and I will forward the remaining books when I have finished them to my satisfaction. I dare say you have not forgotten my telling you that I undertook an investigation of this subject at the request of Naucrates the astronomer, at the time when he came to Alexandria and lived with me, and, when I had worked it out in eight books, I gave them to him at once, too hurriedly, because he was on the point of sailing; they had therefore not been thoroughly revised, indeed I had put down everything just as it occurred to me, postponing revision till the end. Accordingly I now publish, as opportunities serve from time to time, the corrections of the work as they are corrected. In the meantime it has happened that some other persons also, among whom I have met, have got the first and second books as they were corrected; do not be surprised therefore if you come across them in a different shape.

Of the eight books the first four form an elementary introduction. The first contains the modes of producing the sections and the opposite branches (of the hyperbola), the fundamental properties subsisting in them, worked out more fully and generally than in the writings of others. The second book contains the properties of the diameters and axes of the sections as well as the asymptotes, with other properties generally and necessarily used for determining limits of possibility (*διορισμοί*); and what I mean by diameters and axes respectively you will learn from this book. The third book contains many remarkable theorems useful for the syntheses of solid loci and for *diorismi*; the most and the best of these theorems are new, and it was their discovery which made me aware that Euclid did not work out the synthesis of the locus with respect to three and four lines, but only a chance portion of it, and that not successfully; for it was not possible for the said synthesis to be completed without the aid of the additional theorems discovered by me. The fourth book shows in how many ways the sections of cones meet one another and the circumference of a circle; it contains other things in addition, none of which have been treated by earlier writers, namely the questions in how

tions of limits, and the last with determinate conic problems. But of course, when all of them are published, it will be left to all who read them to form their own judgement about them according to their own individual tastes. Farewell.

The preface to Book II merely says that Apollonius is sending the second Book to Eudemus by his son Apollonius and begs Eudemus to communicate it to earnest students of the subject, and in particular to Philonides the geometer whom Apollonius had introduced to Eudemus at Ephesus. There is no preface to Book III as we have it, although the preface to Book IV records that it also was sent to Eudemus.

Preface to Book IV.

Apollonius to Attalus, greeting.

Some time ago I expounded and sent to Eudemus of Pergamum the first three books of my conics which I had compiled in eight books, but, as he has passed away, I have resolved to dedicate the remaining books to you because of your earnest desire to possess my works. I am sending you on this occasion the fourth book. It contains a discussion of the question, in how many points at most it is possible for sections of cones to meet one another and the circumference of a circle, on the assumption that they do not intersect throughout, and further in how many points at most a section of a cone or the circumference of a circle can meet a hyperbola with two branches, [or two double-branch hyperbolas can meet one another]; and, besides these questions, the book considers a number of others of a similar nature. Now the first question Conon expounded to Thrasydaeus of Megara, out, however, showing proper mastery of the proofs on this ground Nicoteles of Cyrene, not without reason, was solved by him. The second matter has merely been mentioned by Nicoteles, in connexion with his controversy with Eratosthenes as one capable of demonstration; but I have not been able to demonstrate either by Nicoteles himself or by any other. The third question and the others akin to it I have not noticed so much as noticed by any one. All the matters required for their solution which I have not found anywhere, required for their solution many and various novel theorems, most of which I have set out as a matter of fact, set out in the first three books, and the rest are contained in the present book. These theorems are of considerable use both for the syntheses of problems

smi. Nicoteles indeed, on account of his controversy with Conon, will not have it that any use can be made of the series of Conon for the purpose of *diorismi*; he is, however, mistaken in this opinion, for, even if it is possible, without using them at all, to arrive at results in regard to questions of possibility, yet they at all events afford a readier means of observing some things, e.g. that several or so many solutions are possible, or again that no solution is possible; such foreknowledge secures a satisfactory basis for investigations, while the theorems in question are again useful in the analyses of *diorismi*. And, even apart from such usefulness, they will be found worthy of acceptance for the sake of the demonstrations themselves, just as we accept of other things in mathematics for this reason and for no other.

The prefaces to Books V–VII now to be given are reproduced for Book V from the translation of L. Nix and for Books VI, VII from that of Halley.

Preface to Book V.

Pollonius to Attalus, greeting.

In this fifth book I have laid down propositions relating to the *maximum* and *minimum* straight lines. You must know that my predecessors and contemporaries have only superficially touched upon the investigation of the shortest lines, and have only proved what straight lines touch the sections conversely, what properties they have in virtue of which they are tangents. For my part, I have proved these propositions in the first book (without however making any use, in the proofs, of the doctrine of the shortest lines), inasmuch as I wished to place them in close connexion with that part of the subject in which I treat of the production of the three sections, in order to show at the same time that in each of the three sections countless properties and necessary results arise, as they do with reference to the original (transverse) diameter. The propositions in which I discuss the shortest lines I have separated into classes, and I have dealt with each individual case by careful demonstration; I have also conducted the investigation of them with the investigation of the greatest lines above mentioned, because I considered that

Preface to Book VI.

Apollonius to Attalus, greeting.

I send you the sixth book of the conics, which embraces propositions about conic sections and segments of conics equal and unequal, similar and dissimilar, besides some other matters left out by those who have preceded me. In particular, you will find in this book how, in a given right cone, a section can be cut which is equal to a given section, and how a right cone can be described similar to a given cone but such as to contain a given conic section. And these matters in truth I have treated somewhat more fully and clearly than those who wrote before my time on these subjects. Farewell.

Preface to Book VII.

Apollonius to Attalus, greeting.

I send to you with this letter the seventh book on conic sections. In it are contained a large number of new propositions concerning diameters of sections and the figures described upon them; and all these propositions have their uses in many kinds of problems, especially in the determination of the limits of their possibility. Several examples of these occur in the determinate conic problems solved and demonstrated by me in the eighth book, which is by way of an appendix and which I will make a point of sending to you as soon as possible. Farewell.

Extent of claim to originality.

We gather from these prefaces a very good idea of the plan followed by Apollonius in the arrangement of the subject and of the extent to which he claims originality. The first four Books form, as he says, an elementary introduction by which he means an exposition of the elements of conics, that is, the definitions and the fundamental propositions which are of the most general use and application; the term 'elements' is in fact used with reference to conics in exactly the same sense as Euclid uses it to describe his great work. The remaining Books beginning with Book V are devoted to more specialized investigation of particular parts of the subject. It is only for a very small portion of the *content* of

ellipse, and circle with the double-branch hyperbola, and of two double-branch hyperbolas with one another, of the investigations which had theretofore only taken account of the single-branch hyperbola. Even in Book V, the most remarkable of all, Apollonius does not say that normals as 'the shortest lines' had not been considered before, but only that they had been superficially touched upon, doubtless in connexion with propositions dealing with the tangent properties. He explains that he found it convenient to treat of the tangent properties, without any reference to normals, in the first Book in order to connect them with the chord properties. It is clear, therefore, that in treating normals as *maxima* and *minima*, and by themselves, without any reference to tangents, as he does in Book V, he was making an innovation; and, in view of the extent to which the theory of normals as maxima and minima is developed by him (in 77 propositions), there is no wonder that he should devote a whole Book to the subject. Apart from the developments in Books III, IV, V, just mentioned, and the numerous new propositions in Book VII with the problems thereon which formed the lost Book VIII, Apollonius only claims to have treated the whole subject more fully and generally than his predecessors.

Great generality of treatment from the beginning.

So far from being a braggart and taking undue credit to himself for the improvements which he made upon his predecessors, Apollonius is, if anything, too modest in his description of his personal contributions to the theory of conic sections. For the 'more fully and generally' of his first preface scarcely conveys an idea of the extreme generality with which the whole subject is worked out. This characteristic generality appears at the very outset.

Analysis of the *Conics*.

Book I.

Apollonius begins by describing a double oblique circular cone in the most general way. Given a circle and any point outside the plane of the circle and in general not lying on the

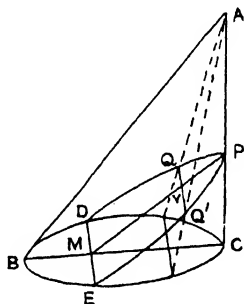
straight line through the centre of the circle perpendicular to its plane, a straight line passing through the point and produced indefinitely in both directions is made to move always passing through the fixed point, so as to pass successively through all the points of the circle; the straight line thus describes a double cone which is in general oblique. Apollonius calls it, *scalene*. Then, before proceeding to the geometry of a cone, Apollonius gives a number of definitions which, though of course only required for conics, are still applicable to any curve.

'In any curve,' says Apollonius, 'I give the name *diameter* to any straight line which, drawn from the curve, bisects all straight lines drawn in the curve (chords) parallel to a given straight line, and I call the extremity of the straight line (i.e. the diameter) which is at the curve a *vertex* of the curve, and each of the parallel straight lines (chords) an *ordinate* (lit. drawn ordinate-wise, *τεταγμένως κατῆχθαι*) to the diameter.'

He then extends these terms to a pair of curves (the pair in reference being to the double-branch hyperbola), giving the name *transverse diameter* to any straight line bisecting all chords in both curves which are parallel to a given straight line (this gives two vertices where the diameter meets the curves respectively), and the name *erect diameter* (*ὀρθή*) to any straight line which bisects all straight lines drawn between one curve and the other which are parallel to a given straight line; the *ordinates* to any diameter are all straight lines parallel straight lines bisected by it. *Conjugate diameters* of any curve or pair of curves are straight lines each of which bisects chords parallel to the other. *Axes* are the principal diameters which cut at right angles the parallel chords which they bisect; and *conjugate axes* are related in the same way as conjugate diameters. Here we have practically our modern definitions, and there is a great advance on Archimedes' terminology.

centre of the circular base. After proving that all sections parallel to the base are also circles, and that there is another set of circular sections subcontrary to these, he proceeds to consider sections of the cone drawn in any other manner. Taking any triangle through the axis (the base of the triangle being consequently a diameter of the circle which is the base of the cone), he is careful to make his section cut the base in a straight line perpendicular to the particular diameter which is the base of the axial triangle. (There is a loss of generality in this, for, if any section is taken, without reference to any axial triangle, we have only to take the particular axial triangle the base of which is that diameter of the circular base which is at right angles to the straight line in which the section of the cone cuts the base.)

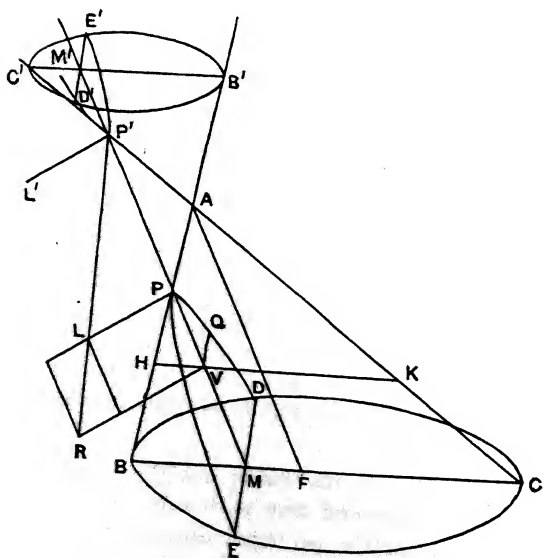
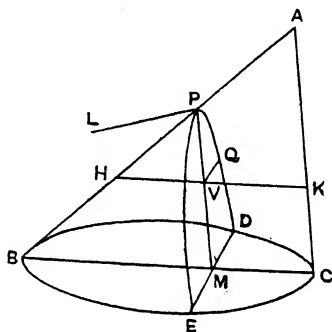
Let ABC be any axial triangle, and let any section whatever cut the cone in a straight line DE at right angles to BC ; if then PM be the intersection of the cutting plane and the plane of the triangle, and if QQ' be any chord of the section parallel to DE , Apollonius proves that QQ' is bisected by PM . In other words, PM is a *diameter* of the section. Apollonius is careful to explain that,



If the cone is a right cone, the straight line in the base (DE) will be at right angles to the common section (PM) of the cutting plane and the triangle through the axis, but, if the cone is scalene, it will not in general be at right angles to PM , but will be at right angles to it only when the plane through the axis (i.e. the axial triangle) is at right angles to the base of the cone' (I. 7).

In other words, Apollonius works out the properties of the conics in the most general way with reference to a diameter which is not one of the principal diameters or axes, but in which the conic has its ordinates obliquely inclined to it. The axes do

as particular cases of the new diameter of reference three sections, the parabola, hyperbola, and ellipse are in the manner shown in the figures. In each case the



$$= BC^2 : BA \cdot AC$$

$$= PL : PA, \text{ by hypothesis}$$

$$= PL : PV : PV : PA,$$

whence

$$QV^2 = PL \cdot PV.$$

(2) In the case of the hyperbola and ellipse,

$$HV : PV = BF : FA,$$

$$VK : P'V = FC : AF.$$

$$\text{Therefore } QV^2 : PV \cdot P'V = HV \cdot VK : PV \cdot P'V$$

$$= BF \cdot FC : AF^2$$

$$= PL : PP', \text{ by hypothesis}$$

$$= RV : P'V$$

$$= PV \cdot VR : PV \cdot P'V,$$

whence

$$QV^2 = PV \cdot VR.$$

New names, 'parabola', 'ellipse', 'hyperbola'.

Accordingly, in the case of the parabola, the square ordinate (QV^2) is equal to the rectangle *applied* to PL with width equal to the abscissa (PV);

in the case of the hyperbola the rectangle applied to PL which is equal to QV^2 and has its width equal to the abscissa PV *overlaps* or *exceeds* ($\upsilon\pi\epsilon\rho\beta\acute{\alpha}\lambda\lambda\epsilon\iota$) by the small rectangle PP' which is similar and similarly situated to the rectangle contained by PL, PP' ;

in the case of the ellipse the corresponding rectangle *short* ($\acute{\epsilon}\lambda\lambda\epsilon\acute{\iota}\pi\epsilon\iota$) by a rectangle similar and similarly situated to the rectangle contained by PL, PP' .

Here then we have the properties of the three curves expressed in the precise language of the Pythagorean application of areas, and the curves are named accordingly: *parabola* ($\pi\alpha\rho\alpha\beta\omicron\lambda\acute{\eta}$) where the rectangle is exactly *applied*, *hyperbola* ($\upsilon\pi\epsilon\rho\beta\omicron\lambda\acute{\eta}$) where it *exceeds*, and *ellipse* ($\acute{\epsilon}\lambda\lambda\epsilon\iota\psi\iota\varsigma$) where it *falls short*.

is called the *latus rectum* (ὀρθία) or the *parameter of ordinates* (παρ' ἣν δύνανται αἱ καταγόμεναι τεταγμένως) in use. In the case of the central conics, the diameter PP' is called the *transverse* (ἡ πλαγία) or *transverse diameter*; while, more commonly, Apollonius speaks of the diameter and corresponding parameter together, calling the latter the *erect side* or *erect side* (ὀρθία πλευρά) and the former the *transverse side* of the figure (εἶδος) *on*, or *applied to*, the figure.

Fundamental properties equivalent to Cartesian equations.

Let p be the parameter, and d the corresponding diameter, the fundamental properties of the curves are the equivalent of the Cartesian equations, referred to the diameter and the tangent at its vertex as axes (in general oblique),

$y^2 = px$ (the parabola),

$y^2 = px \pm \frac{p}{d}x^2$ (the hyperbola and ellipse respectively).

Apollonius expresses the fundamental property of the central conics, like that of the parabola, as an equation in areas, whereas in Archimedes it appears as a proportion.

$$y^2 : (a^2 \pm x^2) = b^2 : a^2,$$

which, however, is equivalent to the Cartesian equation referred to axes with the centre as origin. The latter proposition with reference to the original diameter is separately proved in I. 21, to the effect that QV^2 varies as $PV \cdot P'V$, as is evident from the fact that $QV^2 : PV \cdot P'V = PL : PP'$, and that $PL : PP'$ is constant for any fixed diameter PP' .

Apollonius has a separate proposition (I. 14) to prove that opposite branches of a hyperbola have the same diameter and equal *lata recta* corresponding thereto. As he was unable to treat the double-branch hyperbola fully, he generally speaks of the *hyperbola* (i.e. the single branch) along with

‘If in a hyperbola, an ellipse, or the circumference of a circle, sometimes, however, the double-branch hyperbola and ellipse come in one proposition, e.g. in I. 30: ‘If in an ellipse or the opposites (i.e. the double hyperbola) a straight line is drawn through the centre meeting the curve on both sides of the centre, it will be bisected at the centre.’ The property of conjugate diameters in an ellipse is proved in relation to the original diameter of reference and its conjugate in I. 30 where it is shown that, if DD' is the diameter conjugate to PP' (i.e. the diameter drawn ordinate-wise to PP'), PP' bisects all chords parallel to DD' , so DD' bisects all chords parallel to PP' ; also, if DL' be drawn at right angles to PP' and such that $DL' \cdot DD' = PP'^2$ (or DL' is a third proportional to DD', PP'), then the ellipse has the same property in relation to DD' as diameter and DL' as parameter that it has in relation to PP' as diameter and PL as the corresponding parameter. Incidentally it appears that $PL \cdot PP' = DD'^2$, or PL is a third proportional to PP', DD' , as indeed is obvious from the property of the curve $QV^2 : PV \cdot PV' = PL : PP' = DD'^2$. The next proposition, I. 16, introduces the *secondary diameter* of the double-branch hyperbola (i.e. the diameter conjugate to the transverse diameter of reference), which does not meet the curve; this diameter is defined as that straight line drawn through the centre parallel to the ordinates of the transverse diameter which is bisected at the centre and is of length equal to the mean proportional between the ‘sides of the figure’, i.e. the transverse diameter PP' and the corresponding parameter PL . The *centre* is defined as the middle point of the diameter of reference, and it is proved that all other diameters are bisected at it (I. 30).

Props. 17–19, 22–9, 31–40 are propositions leading to the tangent and containing the tangent properties. On lines exact like those of Eucl. III. 16 for the circle, Apollonius proves that if a straight line is drawn through the vertex (i.e. the extremity of the diameter of reference) parallel to the ordinates of the diameter, it will fall outside the conic, and no other straight line will fall inside the conic.

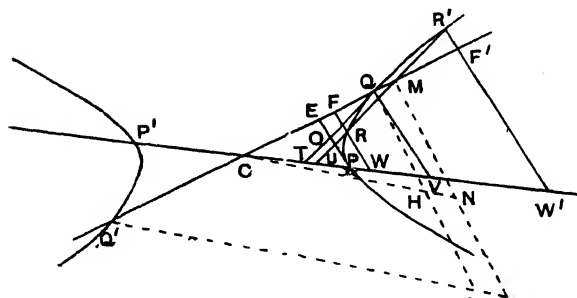
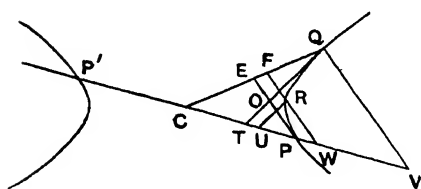
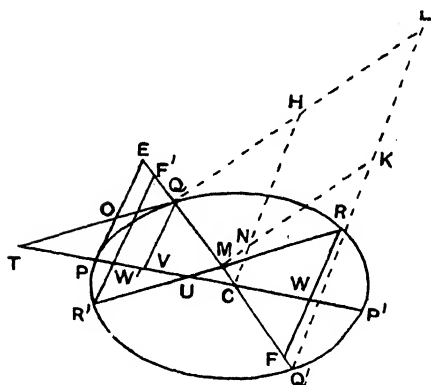
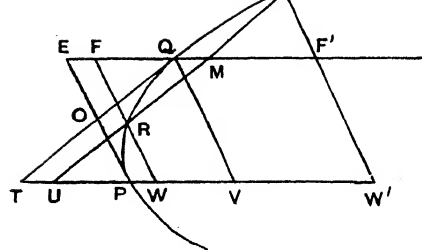
to the original diameter of reference; if Q is the point of contact, QV the ordinate to the diameter through P , and if QT , the tangent at Q , meets the diameter produced in T , then (1) for the parabola $PV = PT$, and (2) for the central conic $TP:TP' = PV:VP'$. The method of proof is to take a point T on the diameter produced satisfying the respective relations, and to prove that, if TQ be joined and produced, any point on TQ on either side of Q is outside the curve: the form of proof is by *reductio ad absurdum*, and in each case it is again proved that no other straight line can fall between TQ and the curve. The fundamental property $TP:TP' = PV:VP'$ for the central conic is then used to prove that $CV \cdot CT = CP^2$ and $QV^2:CV \cdot VT = p:PP'$ (or $CD^2:CP^2$) and the corresponding properties with reference to the diameter DD' conjugate to PP' and v, t , the points where DD' is met by the ordinate to it from Q and by the tangent at Q respectively (Props. I. 37-40).

Transition to new diameter and tangent at its extremity.

An important section of the Book follows (I. 41-50), consisting of propositions leading up to what amounts to a transformation of coordinates from the original diameter and the tangent at its extremity to *any* diameter and the tangent at its extremity; what Apollonius proves is of course that, if *any* other diameter be taken, the ordinate-property of the conic with reference to that diameter is of the same form as it is with reference to the original diameter. It is evident that this is vital to the exposition. The propositions leading up to the result in I. 50 are not usually given in our text-books of geometrical conics, but are useful and interesting.

Suppose that the tangent at any point Q meets the diameter of reference PV in T , and that the tangent at P meets the diameter through Q in E . Let R be any third point on the curve; let the ordinate RW to PV meet the diameter through Q in F , and let RU parallel to the tangent at Q meet PV in U . Then

(1) in the parabola, the triangle $R UW =$ the parallelogram EW , and



(2) in the hyperbola or ellipse, $\Delta RUW =$ the difference between the triangles CFW and CPE .

$$\begin{aligned} (1) \text{ In the parabola } \Delta RUW : \Delta QTV &= RW^2 : QV^2 \\ &= PW : PV \\ &= \square EW : \square EV. \end{aligned}$$

But, since $TV = 2PV$, $\Delta QTV = \square EV$;

therefore $\Delta RUW = \square EW$.

(2) The proof of the proposition with reference to the central conic depends on a Lemma, proved in I. 41, to the effect that, if PX , VY be similar parallelograms on CP , CV as bases, and if VZ be an equiangular parallelogram on QV as base and such that, if the ratio of CP to the other side of PX is m , the ratio of QV to the other side of VZ is $m \cdot p / PP'$, then VZ is equal to the difference between VY and PX . The proof of the Lemma by Apollonius is difficult, but the truth of it can be easily seen thus.

By the property of the curve, $QV^2 : CV^2 \sim CP^2 = p : PP'$;

$$\text{therefore } CV^2 \sim CP^2 = \frac{PP'}{p} \cdot QV^2.$$

Now $\square PX = \mu \cdot CP^2 / m$, where μ is a constant depending on the angle of the parallelogram.

Similarly

$$\square VY = \mu \cdot CV^2 / m, \text{ and } \square VZ = \mu \cdot \frac{PP'}{p} QV^2 / m.$$

It follows that $\square VY \sim \square PX = \square VZ$.

Taking now the triangles CFW , CPE and RUW in the ellipse or hyperbola, we see that CFW , CPE are similar, and RUW has one angle (at W) equal or supplementary to the angles at P and V in the other two triangles, while we have

$$QV^2 : CV \cdot VT = p : PP',$$

therefore

$$\Delta RUW = \Delta CFW \sim \Delta CPE.$$

The same property with reference to the diameter *sec* to *CPV* is proved in I. 45.

It is interesting to note the exact significance of the property thus proved for the central conic. The proposition, which is the foundation of Apollonius's method of transformation of coordinates, amounts to this. If *CP*, *CQ* are fixed diameters and *R* a variable point, the area of the quadrilateral *CFRU* is constant for all positions of *R* on the conic. Suppose now that *CP*, *CQ* are taken as axes of *x* and *y* respectively. If we draw *RX* parallel to *CQ* to meet *CP* and *RY* parallel to *CP* to meet *CQ*, the proposition asserts that (subject to proper convention as to sign)

$$\Delta RYF + \square CXRY + \Delta RXU = (\text{const.}).$$

But since *RX*, *RY*, *RF*, *RU* are in fixed directions,

ΔRYF varies as RY^2 or x^2 , $\square CXRY$ as $RX \cdot RY$ and ΔRXU as RX^2 or y^2 .

Hence, if *x*, *y* are the coordinates of *R*,

$$\alpha x^2 + \beta xy + \gamma y^2 = A,$$

which is the Cartesian equation of the conic referred to its centre as origin and any two diameters as axes.

The properties so obtained are next used to prove that if *UR* meets the curve again in *R'* and the diameter *CP* in *M*, then *RR'* is bisected at *M*. (I. 46-8).

Taking (1) the case of the parabola, we have,

$$\Delta RUW = \square EW,$$

and

$$\Delta R'UW' = \square EW'.$$

By subtraction, $(RWW'R') = \square F'W,$

whence

$$\Delta RFM = \Delta R'F'M,$$

and, since the triangles are similar, $RM = R'M$.

The same result is easily obtained for the central conic.

It follows that *EQ* produced in the case of the parabola

or CQ in the case of the central conic, bisects all chords as RR' parallel to the tangent at Q . Consequently EQ and CQ are *diameters* of the respective conics.

In order to refer the conic to the new diameter and the corresponding ordinates, we have only to determine the *parameter* of these ordinates and to show that the property of the conic with reference to the new parameter and diameter is in the same form as that originally found.

The propositions I. 49, 50 do this, and show that the new parameter is in all the cases p' , where (if O is the point of intersection of the tangents at P and Q)

$$OQ : QE = p' : 2QT.$$

(1) In the case of the parabola, we have $TP = PV = EQ$,

whence

$$\triangle EOQ = \triangle POT.$$

Add to each the figure $POQF'W'$;

therefore

$$QTW'F' = \square EW' = \triangle R'UW',$$

whence, subtracting $MUW'F'$ from both, we have

$$\triangle R'MF' = \square QU.$$

Therefore

$$R'M \cdot MF' = 2QT \cdot QM.$$

But $R'M : MF' = OQ : QE = p' : 2QT$, by hypothesis;

therefore $R'M^2 : R'M \cdot MF' = p' \cdot QM : 2QT \cdot QM$.

And $R'M \cdot MF' = 2QT \cdot QM$, from above;

therefore

$$R'M^2 = p' \cdot QM,$$

which is the desired property.¹

¹ The proposition that, in the case of the parabola, if p be the parameter of the ordinates to the diameter through Q , then (see the first figure on p. 142)

$$OQ : QE = p : 2QT$$

(2) In the case of the central conic, we have

$$\Delta R'UW' = \Delta CF'W' \sim \Delta CPE.$$

(Apollonius here assumes what he does not prove till 1 namely that $\Delta CPE = \Delta CQT$. This is proved thus.

We have $CV:CT = CV^2:CP^2$; (I. 3)

therefore $\Delta CQV:\Delta CQT = \Delta CQV:\Delta CPE$,

so that $\Delta CQT = \Delta CPE$.)

Therefore $\Delta R'UW' = \Delta CF'W' \sim \Delta CQT$,

and it is easy to prove that in all cases

$$\Delta R'MF' = \Delta QTUM.$$

Therefore $R'M.MF' = QM(QT + MU)$.

Let QL be drawn at right angles to CQ and equal CQ . Join $Q'L$ and draw MK parallel to QL to meet $Q'L$. Draw CH parallel to $Q'L$ to meet QL in H and MK in N .

Now $R'M:MF' = OQ:QE$

$= QL:2QT$, by hypothesis,

$= QH:QT$.

But $QT:MU = CQ:CM = QH:MN$,

so that $(QH + MN):(QT + MU) = QH:QT$

$= R'M:MF'$, from above

where p_a is the parameter of the principal ordinates and p^* the parameter of the ordinates to the diameter PV .

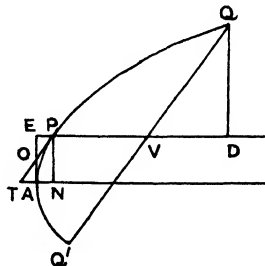
If the tangent at the vertex A meets VP produced in E , and PT , the tangent at P , in O , the proposition of Apollonius proves that

$$OP:PE = p:2PT.$$

But $OP = \frac{1}{2}PT$;

therefore $PT^2 = p \cdot PE$
 $= p \cdot AN$.

Thus $QV^2:QD^2 = PT^2:PN^2$, by similar triangles,
 $= p \cdot AN:p_a \cdot AN$



follows that

$$QM(QH + MN) : QM(QT + MU) = R'M^2 : R'M \cdot MF';$$

from above, $QM(QT + MU) = R'M \cdot MF';$

therefore $R'M^2 = QM(QH + MN)$

$$= QM \cdot MK,$$

which is the desired property.

In the case of the hyperbola, the same property is true for the opposite branch.

These important propositions show that the ordinate property of the three conics is of the same form whatever diameter is taken as the diameter of reference. It is therefore a matter of no difference to which particular diameter and ordinates the property is referred. This is stated by Apollonius in a summary which follows I. 50.

First appearance of principal axes.

The *axes* appear for the first time in the propositions next following (I. 52–8), where Apollonius shows how to construct the conics, given in each case (1) a diameter, (2) the length of the corresponding parameter, and (3) the inclination of the ordinates to the diameter. In each case Apollonius assumes the angle between the ordinates and the diameter to be a right angle; then he reduces the case where the angle is oblique to the case where it is right by his method of translation of coordinates; i.e. from the given diameter and parameter he *finds* the *axis* of the conic and the length of the corresponding parameter, and he then constructs the conic as in the first case where the ordinates are at right angles to the diameter. Here then we have a case of the proof of *existence* means of *construction*. The conic is in each case constructed by finding the cone of which the given conic is a section. The problem of finding the axis of a parabola and

account of Book I, to show not merely what is obtained by Apollonius, but the way in which his work; and it will have been realized how entirely and general the method is. When the foundation is laid and the fundamental properties established, Apollonius develops the rest of the subject on lines more or less those followed in our text-books. My description of the work can therefore for the most part be considered a summary of the contents.

Book II begins with a section devoted to the properties of the asymptotes. They are constructed in II. 1 in the following manner. Beginning, as usual, with *any* diameter of reference and its corresponding parameter and inclination of ordinates, Apollonius draws at P the vertex (the extremity of the transverse axis) a tangent to the hyperbola and sets off along it lengths on either side of P such that $PL^2 = PL'^2 = \frac{1}{4}p \cdot PL$, where p is the parameter. He then proves that the lines CL and CL' produced will not meet the curve in any finite point and are therefore *asymptotes*. II. 2 proves further that no straight line through C within the angle between the asymptotes can be an asymptote. II. 3 proves that the intercept made by the asymptotes on the tangent at any point P is bisected at P ; that the square on each half of the intercept is equal to one-fourth of the 'figure' corresponding to the diameter CP (i.e. one-fourth of the rectangle contained by the transverse side, the *latus rectum* or parameter corresponding to the diameter, and the diameter itself); this property is used as a means of drawing a hyperbola when the asymptotes and a point on the curve are given (II. 4). II. 5-7 are devoted to properties about a tangent at the extremity of a diameter being perpendicular to the chords bisected by it. Apollonius returns to the asymptotes in II. 8, and II. 8-14 give the other properties with reference to the asymptotes (II. 9 is the converse of II. 3), the equality of the intercepts between the asymptotes and the curve of any chord (II. 8), the equality of the rectangle contained by the distances between the asymptotes in which the chord meets the curve and the point of intersection with the asymptotes to the square on the semi-diameter (II. 10), the latter property with re-

the portions of the asymptotes which include between them a branch of the conjugate hyperbola (II. 11), the constancy of the rectangle contained by the straight lines drawn from any point of the curve in fixed directions to meet the asymptotes (equivalent to the Cartesian equation with reference to the asymptotes, $xy = \text{const.}$) (II. 12), and the fact that the curve and the asymptotes proceed to infinity and approach continually nearer to one another, so that the distance separating them can be made smaller than any given length (II. 14). II. 15 proves that the two opposite branches of a hyperbola have the same asymptotes and II. 16 proves for the chord connecting points on two branches the property of II. 8. II. 17 shows that 'conjugate opposites' (two conjugate double-branch hyperbolas) have the same asymptotes. Propositions follow about conjugate hyperbolas; any tangent to the conjugate hyperbola will meet both branches of the original hyperbola and will be bisected at the point of contact (II. 19); if Q be any point on a hyperbola, and CE parallel to the tangent at Q meets the conjugate hyperbola in E , the tangent at E will be parallel to CQ and CQ, CE will be conjugate diameters (II. 20), while the tangents at Q, E will meet on one of the asymptotes (II. 21); if a chord Qq in one branch of a hyperbola meet the asymptotes in R, r and the conjugate hyperbola in Q', q' , then $Q'Q \cdot Qq' = 2CD^2$ (II. 23). Of the rest of the propositions in this part of the Book the following may be mentioned: if TQ, TQ' are two tangents to a conic and V is the middle point of QQ' , TV is a diameter (II. 29, 30, 38); if tQ, tQ' be tangents to opposite branches of a hyperbola, RR' the chord through t parallel to QQ' , v the middle point of QQ' , then vR, vR' are tangents to the hyperbola (II. 40); in a conic, or a circle, or in conjugate hyperbolas, if two chords not passing through the centre intersect, they do not bisect each other (II. 26, 41, 42). II. 44-7 show how to find a diameter of a conic and the centre of a central conic, the axis of a parabola and the axes of a central conic. The Book concludes with problems of drawing tangents to conics in

the last problem, proving that, if the tangent to an ellipse at any point P meets the major axis in T , the angle CPT is greater than the angle ABA' , where B is one extremity of the minor axis.

Book III begins with a series of propositions about the equality of certain areas, propositions of the same kind as are easily derived from, the propositions (I. 41-50) by means of which, as already shown, the transformation of coordinates is effected. We have first the proposition that, if the tangents at any points P, Q of a conic meet in O , and if they meet the diameters through Q, P respectively in E, T , then $\triangle OPT = \triangle OQE$ (III. 1, 4); and, if P, Q be points on adjacent branches of conjugate hyperbolas, $\triangle CPE = \triangle CQT$ (III. 2). With the same notation, if R be any other point on the conic and if we draw RU parallel to the tangent at Q meeting the diameter through P in U and the diameter through Q in H , and RW parallel to the tangent at P meeting QT in I , then the quadrilaterals $HQF, HTUR$ are equal, where F is the intersection of the diameters through Q, P in F, W , then $\triangle HQF = \triangle HTUR$ (III. 2, 6); this is proved at once from the fact that $\triangle RMF = \text{quadrilateral } QTUM$ (see I. 49, 50, or pp. 100, 101 above) by subtracting or adding the area $HRMQ$ on the same side. Next take any other point R' , and draw $R'U', F'H'$ in the same way as before; it is then proved that, if $RU, R'U'$ meet in I and $R'U', RW$ in J , the quadrilaterals $F'IRF, IUUR$ are equal, and also the quadrilaterals $FJR'F', JU'UR$ (III. 7, 9, 10). The proof varies according to the actual position of the points in the figures.

In Figs. 1, 2 $\triangle HFQ = \text{quadrilateral } HTUR$,

$$\triangle H'F'Q = H'TU'R'.$$

By subtraction, $FHH'F' = IUU'R' \mp (IH)$;

whence, if IH be added or subtracted, $F'IRF = IUU'R$

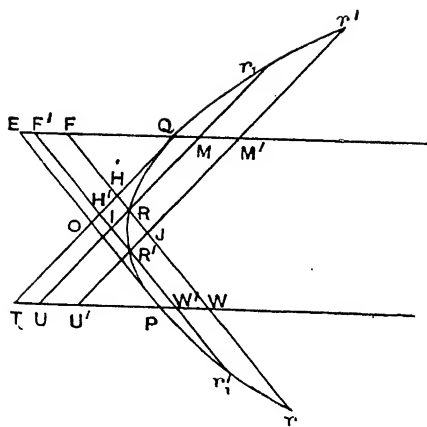


FIG. 1.

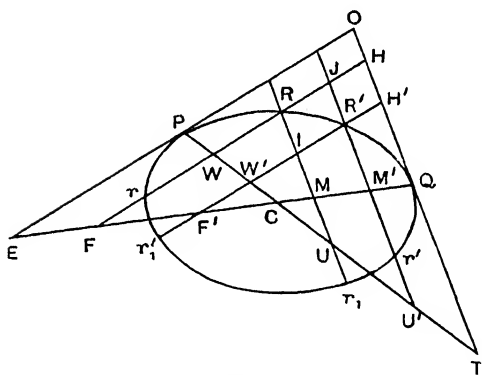
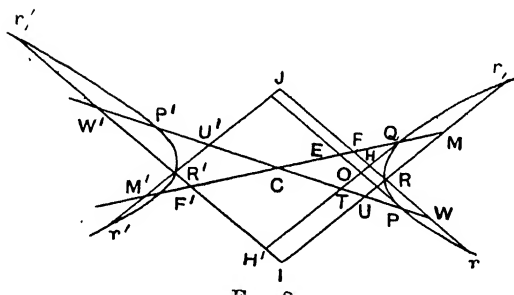


FIG. 2.



Adding the quadrilateral $CF'H'T$, we have

$$\Delta H'F'Q = H'TU'R',$$

and similarly

$$\Delta HFQ = HTUR.$$

By subtraction, $F'H'HF = H'TU'R' - HTUR$.

Adding $H'IRH$ to each side, we have

$$F'IRF = IUU'R'.$$

If each of these quadrilaterals is subtracted from IJ ,

$$FJR'F' = JU'UR.$$

The corresponding results are proved in III. 5, 11, 12 for the case where the ordinates through RR' are drawn a *secondary* diameter, and in III. 15 for the case where they are on the original hyperbola and R, R' on the conjugate hyperbola.

The importance of these propositions lies in the fact that they are immediately used to prove the well-known theorems about the rectangles contained by the segments of intersecting chords and the harmonic properties of the pole and polar. The former question is dealt with in III. 16-23, which treat a great variety of particular cases. We will give the statement of one case, to the effect that, if OP, OQ be two tangents to any conic and $Rr, R'r'$ be any two chords parallel to them respectively and intersecting in J , an internal or external point,

then $RJ \cdot Jr : R'J \cdot Jr' = OP^2 : OQ^2 = (\text{const.})$.

We have

$$RJ \cdot Jr = RW^2 \sim JW^2, \text{ and } RW^2 : JW^2 = \Delta RUW : \Delta JUW$$

therefore

Similarly $R'M'^2 : JM'^2 = \Delta R'F'M' : \Delta JFM'$,

whence $R'J \cdot Jr' : R'M'^2 = FJR'F' : \Delta R'F'M'$.

But $R'M'^2 : OQ^2 = \Delta R'F'M' : \Delta OQE$;

therefore, *ex aequali*, $R'J \cdot Jr' : OQ^2 = FJR'F' : \Delta OQE$.

It follows, since $FJR'F' = JU'UR$, and $\Delta OPT = \Delta OQE$,

that $RJ \cdot Jr : OP^2 = R'J \cdot Jr' : OQ^2$,

or $RJ \cdot Jr : R'J \cdot Jr' = OP^2 : OQ^2$.

If we had taken chords Rr_1 , $R'r'_1$ parallel respectively to OQ , OP and intersecting in I , an internal or external point, we should have in like manner

$$RI \cdot Ir_1 : R'I \cdot Ir'_1 = OQ^2 : OP^2.$$

As a particular case, if PP' be a diameter, and Rr , $R'r'$ be chords parallel respectively to the tangent at P and the diameter PP' and intersecting in I , then (as is separately proved)

$$RI \cdot Ir : R'I \cdot Ir' = p : PP'.$$

The corresponding results are proved in the cases where certain of the points lie on the conjugate hyperbola.

The six following propositions about the segments of intersecting chords (III. 24–9) refer to two chords in conjugate hyperbolas or in an ellipse drawn parallel respectively to two conjugate diameters PP' , DD' , and the results in modern form are perhaps worth quoting. If Rr , $R'r'$ be two chords so drawn and intersecting in O , then

(a) in the conjugate hyperbolas

$$\frac{RO \cdot Or}{CP^2} \pm \frac{R'O \cdot Or'}{CD^2} = 2,$$

and $(RO^2 + Or^2) : (R'O^2 + Or'^2) = CP^2 : CD^2$;

The general propositions containing the harmonic properties of the pole and polar of a conic are III. 37-40, which prove that in any conic, if TQ , Tq be tangents, and if Qq the chord of contact be bisected in V , then

(1) if any straight line through T meet the conic in R' , R and Qq in I , then (Fig. 1) $RT:TR' = RI:IR'$;

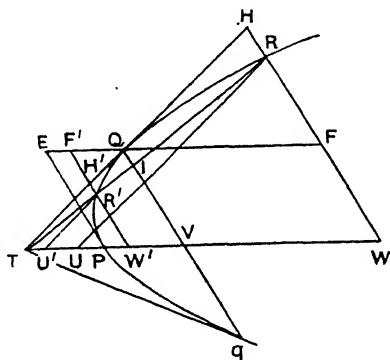
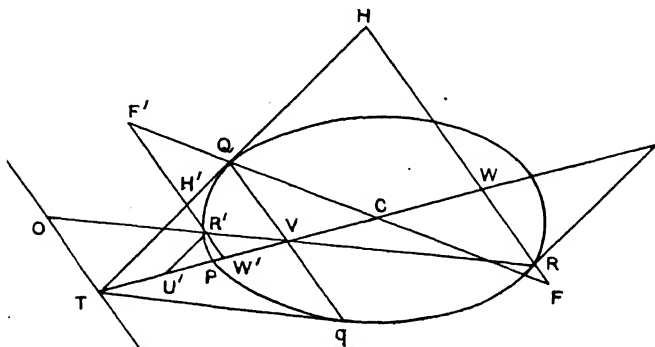


FIG. 1.

(2) if any straight line through V meet the conic in R , R' and the parallel through T to Qq in O , then (Fig. 2)

$$RO:OR' = RV:VR'.$$



To prove (1) we have

$$R'I^2:IR^2 = H'Q^2:QH^2 = \Delta H'F'Q:\Delta HFQ = H'TU'R':HTUR \\ \text{(III. 2, 3, \&c.).}$$

$$\text{Also } R'T^2:TR^2 = R'U'^2:UR^2 = \Delta R'U'W':\Delta RUW,$$

$$\text{and } R'T^2:TR^2 = TW'^2:TW^2 = \Delta TH'W':\Delta THW,$$

$$\text{so that } R'T^2:TR^2 = \Delta TH'W' \sim \Delta R'U'W':\Delta THW \sim \Delta RUW \\ = H'TU'R':HTUR \\ = R'I^2:IR^2, \text{ from above.}$$

To prove (2) we have

$$RV^2:VR^2 = RU^2:R'U'^2 = \Delta RUW:\Delta R'U'W',$$

and also

$$= HQ^2:QH'^2 = \Delta HFQ:\Delta H'F'Q = HTUR*:H'TU'R',$$

so that

$$RV^2:VR^2 = HTUR \pm \Delta RUW:H'TU'R' \pm \Delta R'U'W' \\ = \Delta THW:\Delta TH'W' \\ = TW^2:TW'^2 \\ = RO^2:OR'^2.$$

Props. III. 30–6 deal separately with the particular cases in which (a) the transversal is parallel to an asymptote of the hyperbola or (b) the chord of contact is parallel to an asymptote, i.e. where one of the tangents is an asymptote, which is the tangent at infinity.

Next we have propositions about intercepts made by two tangents on a third: If the tangents at three points of a parabola form a triangle, all three tangents will be cut by the points of contact in the same proportion (III. 41); if the tangents at the extremities of a diameter PP' of a central conic are cut in α, α' by any other tangent PP' , $PP' = CD^2$ (III. 42).

L, L' and M, M' respectively, then $L'M, LM'$ are both pa
to PQ (III. 44).

The first of these propositions asserts that, if the tange
three points P, Q, R of a parabola form a triangle pqr , th

$$Pr:rq = rQ:Qp = qp:pR.$$

From this property it is easy to deduce the Cart
equation of a parabola referred to two fixed tangen
coordinate axes. Taking qR, qP as fixed coordinate axe
find the locus of Q thus. Let x, y be the coordinates
Then, if $qp = x_1, qr = y_1, qR = h, qP = k$, we have

$$\frac{x}{x_1 - x} = \frac{rQ}{Qp} = \frac{y_1 - y}{y} = \frac{k - y_1}{y_1} = \frac{x_1}{h - x_1}.$$

From these equations we derive

$$x_1^2 = hx, y_1^2 = ky;$$

also, since $\frac{x_1}{x} = \frac{y_1}{y_1 - y}$, we have $\frac{x}{x_1} + \frac{y}{y_1} = 1$.

By substituting for x_1, y_1 the values \sqrt{hx}, \sqrt{ky}
obtain

$$\left(\frac{x}{h}\right)^{\frac{1}{2}} + \left(\frac{y}{k}\right)^{\frac{1}{2}} = 1.$$

The focal properties of central conics are prove
III. 45-52 without any reference to the directrix; the
no mention of the focus of a parabola. The foci are c
'the points arising out of the application' ($\tau\acute{\alpha} \epsilon\kappa \tau\eta\varsigma \pi$
 $\beta\omicron\lambda\eta\varsigma \gamma\iota\nu\acute{o}\mu\epsilon\nu\alpha \sigma\eta\mu\epsilon\iota\alpha$), the meaning being that S, S' are t
on the axis AA' such that $AS.SA' = AS'.S'A' = \frac{1}{4}p_a$
or CB^2 , that is, in the phraseology of application of a
a rectangle is applied to AA' as base equal to one-f
part of the 'figure', and in the case of the hyperbola
ceeding, but in the case of the ellipse falling short, l
square figure. The foci being thus found, it is proved
if the tangents $Ar, A'r'$ at the extremities of the axis are
by the tangent at any point D in a 'conic'.

used to prove that the focal distances of P make equal angles with the tangent at P (III. 48). In III. 49–52 follow the other ordinary properties, that, if SY be perpendicular to the tangent at P , the locus of Y is the circle on AA' as diameter, that the lines from C drawn parallel to the focal distances to meet the tangent at P are equal to CA , and that the sum or difference of the focal distances of any point is equal to AA' .

The last propositions of Book III are of use with reference to the locus with respect to three or four lines. They are as follows.

1. If PP' be a diameter of a central conic, and if $PQ, P'Q$ drawn to any other point Q of the conic meet the tangents at P', P in R', R respectively, then $PR \cdot P'R' = 4CD^2$ (III. 53).

2. If TQ, TQ' be two tangents to a conic, V the middle point of QQ' , P the point of contact of the tangent parallel to QQ' , and R any other point on the conic, let Qr parallel to TQ' meet $Q'R$ in r , and $Q'r'$ parallel to TQ meet QR in r' ; then

$$Qr : Q'r' : QQ'^2 = (PV^2 : PT^2) \cdot (TQ \cdot TQ' : QV^2). \quad (\text{III. 54, 56.})$$

3. If the tangents are tangents to opposite branches of a hyperbola and meet in t , and if R, r, r' are taken as before, while tq is half the chord through t parallel to QQ' , then

$$Qr \cdot Q'r' : QQ'^2 = tq : tQ' : tq^2. \quad (\text{III. 55.})$$

The second of these propositions leads at once to the three-line locus, and from this we easily obtain the Cartesian equation to a conic with reference to two fixed tangents as axes, where the lengths of the tangents are h, k , viz.

$$\left(\frac{x}{h} + \frac{y}{k} - 1\right)^2 = 2\lambda \left(\frac{xy}{hk}\right)^{\frac{1}{2}}.$$

Book IV is on the whole dull, and need not be noticed at length. Props. 1–23 prove the converse of the propositions in Book III about the harmonic properties of the pole and polar

tively. Take O on QQ' and O' on RR' so that TQ', TR' harmonically divided. The intersections of OO' produced with the conic give the two points of contact required.

The remainder of the Book (IV.24–57) deals with intersection of conics, and the number of points in which, in particular cases, they can intersect or touch. IV. 24 proves that no two conics can meet in such a way that part of one of them is common to both, while the rest is not. The rest of the propositions can be divided into five groups, three of which can be brought under one general enunciation. Group I consists of particular cases depending on the more elementary considerations affecting conics: e.g. two conics having their concavities in opposite directions will not meet in more than two points (IV. 25); if a conic meet one branch of a hyperbola, it will not meet the other branch in more points than two (IV. 37); a conic touching one branch of a hyperbola with its concave side will not meet the opposite branch (IV. 39). IV. 36, 41, 42, 54 belong to this group. Group II contains propositions (IV. 25, 38, 43, 44, 46, 55) showing that no two conics (including in the term the double-branch hyperbola) intersect in more than four points. Group III (IV. 26, 47, 49, 50, 56) are particular cases of the proposition that two conics which touch at one point cannot intersect at more than two other points. Group IV (IV. 27, 28, 29, 40, 51, 52, 53, 57) are cases of the proposition that no two conics which touch each other at two points can intersect at any other point. Group V consists of propositions about double contact. Two parabolas cannot touch another parabola in more points than one (IV. 30); this follows from the property $TP = PV$. A parabola, if it fall outside a hyperbola, cannot have double contact with it (IV. 31); it is shown that for the hyperbola $PV > PT$, while for the parabola $P'V = P'T$; therefore a hyperbola would fall outside the parabola, which is impossible. A parabola cannot have internal double contact with an ellipse or circle (IV. 32). A hyperbola cannot have double contact with another hyperbola having the same centre (IV. 33).

remarkable of the extant Books. It deals with normals to conics regarded as *maximum* and *minimum* straight lines drawn from particular points to the curve. Included in it are series of propositions which, though worked out by the most geometrical methods, actually lead immediately to the determination of the evolute of each of the three conics; that is to say, the Cartesian equations to the evolutes can be easily deduced from the results obtained by Apollonius. There can be no doubt that the Book is almost wholly original, and it is a valuable geometrical *tour de force*.

Apollonius in this Book considers various points and classes of points with reference to the maximum or minimum straight line which it is possible to draw from them to the conics, and the feet of normals to the curve. He begins naturally with points on the axis, and he takes first the point E where measured along the axis from the vertex A is $\frac{1}{2}p$, p being the principal parameter. The first three propositions prove generally and for certain particular cases that, if in an ellipse or hyperbola AM be drawn at right angles to AA' and equal to AA' , and if CM meet the ordinate PN of any point P of the curve in H , then $PN^2 = 2$ (quadrilateral $MANH$); this is a lemma used in the proofs of later propositions, V. 5, 6, &c. In V. 4, 5, 6, he proves that, if $AE = \frac{1}{2}p$, then AE is the *minimum* straight line from E to the curve, and if P be any point on it, PE increases as P moves farther away from E on either side; he proves in fact that, if PN be the ordinate of P ,

in the case of the parabola $PE^2 = AE^2 + AN^2$,

in the case of the hyperbola or ellipse

$$PE^2 = AE^2 + AN^2 \cdot \frac{AA' \pm p}{AA'},$$

where of course $p = BB'^2/AA'$, and therefore $(AA' \pm p)/AA'$ is equivalent to what we call e^2 , the square of the eccentricity. Apollonius also proved that EA' is the *maximum* straight line from

Next Apollonius takes points G on the axis at a distance from A greater than $\frac{1}{2}p$, and he proves that the *minimum* straight line from G to the curve (i.e. the normal) is where P is such a point that

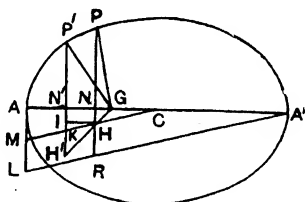
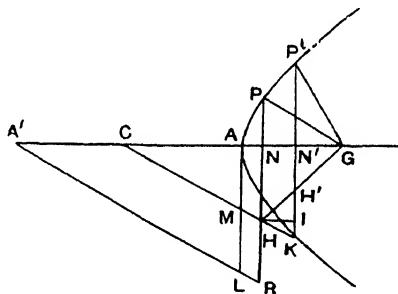
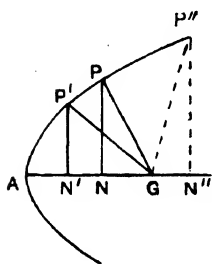
(1) in the case of the parabola $NG = \frac{1}{2}p$;

(2) in the case of the central conic $NG:CN = p:AA'$;

and, if P' is any other point on the conic, $P'G$ increases as P moves away from P' on either side; this is proved by showing that

(1) for the parabola $P'G^2 = PG^2 + NN'^2$;

(2) for the central conic $P'G^2 = PG^2 + NN'^2 \cdot \frac{AA' \pm p}{AA'}$.



As these propositions contain the fundamental properties of the subnormals, it is worth while to reproduce Apollonius' proofs.

We have $P'N'^2 = p \cdot AN' = 2NG \cdot AN'$;
 and $N'G^2 = NN'^2 + NG^2 \pm 2NG \cdot NN'$,
 according to the position of N' .

$$\begin{aligned}\text{Therefore } P'G^2 &= 2NG \cdot AN + NG^2 + NN'^2 \\ &= PN^2 + NG^2 + NN'^2 \\ &= PG^2 + NN'^2;\end{aligned}$$

and the proposition is proved.

(2) In the case of the central conic, take G on the axis such that $AG > \frac{1}{2}p$, and measure GN towards A such that

$$NG : CN = p : AA'.$$

Draw the ordinate PN through N , and also the ordinate $P'N'$ from any other point P' .

We have first to prove the lemma (V. 1, 2, 3) that, if AM be drawn perpendicular to AA' and equal to $\frac{1}{2}p$, and if CM , produced if necessary, meet PN in H , then

$$PN^2 = 2(\text{quadrilateral } MANH).$$

This is easy, for, if $AL (= 2AM)$ be the parameter, and $A'L$ meet PN in R , then, by the property of the curve,

$$\begin{aligned}PN^2 &= AN \cdot NR \\ &= AN(NH + AM) \\ &= 2(\text{quadrilateral } MANH).\end{aligned}$$

Let GH , produced if necessary, meet $P'N'$ in H' . From H draw HI perpendicular to $P'H'$.

$$\begin{aligned}\text{Now, since, by hypothesis, } NG : CN &= p : AA' \\ &= AM : AC \\ &= HN : NC,\end{aligned}$$

$NH = NG$, whence also $H'N' = N'G$.

$$\begin{aligned}
P'G^2 &= N'G^2 + P'N'^2 \\
&= 2\Delta H'N'G + 2(\Delta MKN') \\
&= 2(\Delta MHG) + 2\Delta HH'K.
\end{aligned}$$

Therefore, by subtraction,

$$\begin{aligned}
P'G^2 - PG^2 &= 2\Delta HH'K \\
&= HI \cdot (H'I \pm IK) \\
&= HI \cdot (HI \pm IK) \\
&= HI^2 \cdot \frac{CA \pm AM}{CA} \\
&= NN'^2 \cdot \frac{AA' \pm p}{AA'};
\end{aligned}$$

which proves the proposition.

If O be any point on PG , OP is the minimum straight line from O to the curve, and OP' increases as P' moves from P on either side; this is proved in V. 12. (Since $\angle GPP' > \angle GP'P$; therefore, *a fortiori*, $\angle OPP' > \angle OP'P$ and $OP' > OP$.)

Apollonius next proves the corresponding proposition with reference to points on the *minor* axis of an ellipse. Let p' be the parameter of the ordinates to the minor axis, $p' = \frac{1}{2}p = CA^2/CB$. If now E' be so taken that BE' is the *maximum* straight line from E' to the curve, and, if P be any other point on it, $E'P$ diminishes as P moves farther from B on either side, and $E'B'$ is the *minimum* straight line from E' to the curve. It is, in fact,

$$E'B^2 - E'P^2 = Bn^2 \cdot \frac{p' - BB'}{BB'}, \text{ where } Bn \text{ is the abscissa.}$$

(V. 16-18). If O be any point on the minor axis, $BO > BE'$, then OB is the *maximum* straight line from O to the curve, &c. (V. 19).

If g be a point on the minor axis such that $Bg < \frac{1}{2}p'$, and if Cn be measured towards B so that

$$Cn : ng = BB' : p',$$

then n is the foot of the ordinates of two points on the curve. Pg is the *maximum* straight line from g to the curve.

be any other point on it, $P'g$ diminishes as P' moves
er from P on either side to B or B' , and

$$Pg^2 - P'g^2 = nn'^2 \cdot \frac{p' - BB'}{BB'} \quad \text{or} \quad nn'^2 \cdot \frac{CA^2 - CB^2}{CB^2}.$$

be any point on Pg produced beyond the minor axis, PO
maximum straight line from O to the same part of the
e for which Pg is a maximum, i.e. the semi-ellipse BPB' ,
(V. 20-2).

V. 23 it is proved that, if g is on the minor axis, and gP
imum straight line to the curve, and if Pg meets AA'
then GP is the *minimum* straight line from G to the
; this is proved by similar triangles. Only one normal
e drawn from any one point on a conic (V. 24-6). The
al at any point P of a conic, whether regarded as a
um straight line from G on the major axis or (in the
f the ellipse) as a *maximum* straight line from g on the
axis, is perpendicular to the tangent at P (V. 27-30);
neral (1) if O be any point within a conic, and OP be
imum or a minimum straight line from O to the conic,
straight line through P perpendicular to PO touches the
and (2) if O' be any point on OP produced outside the
 $O'P$ is the minimum straight line from O' to the conic,
(V. 31-4).

Number of normals from a point.

now come to propositions about two or more normals
ng at a point. If the normal at P meet the axis of
abola or the axis AA' of a hyperbola or ellipse in G , the
 PGA increases as P or G moves farther away from A ,
a the case of the hyperbola the angle will always be less
the complement of half the angle between the asymptotes.
normals at points on the same side of AA' will meet on
opposite side of that axis; and two normals at points on
ame quadrant of an ellipse as AB will meet at a point

and others not (V. 41-3).

If P_1G_1 , P_2G_2 be normals at points on one side of the a conic meeting in O , and if O be joined to any other on the conic (it being further supposed in the case ellipse that all three lines OP_1 , OP_2 , OP cut the same the axis), then

- (1) OP cannot be a normal to the curve;
- (2) if OP meet the axis in K , and PG be the normal at P is less or greater than AK according as P does or does between P_1 and P_2 .

From this proposition it is proved that (1) three normals at points on one quadrant of an ellipse cannot meet at one point and (2) four normals at points on one semi-ellipse bounded by the major axis cannot meet at one point (V. 44-8).

In any conic, if M be any point on the axis such that AM is not greater than $\frac{1}{2}p$, and if O be any point on the curve, the ordinate through M , then no straight line drawn to any point on the curve on the other side of the axis from O and the axis between A and M can be a normal (V. 49, 50).

Propositions leading immediately to the determination of the evolute of a conic.

These great propositions are V. 51, 52, to the following effect:

If AM measured along the axis be greater than $\frac{1}{2}p$ in the case of the ellipse less than AC), and if MO be drawn perpendicular to the axis, then a certain length (y , say), may be assigned such that

- (a) if $OM > y$, no normal can be drawn through O which cuts the axis; but, if OP be any straight line drawn to the curve cutting the axis in K , $NK < NG$, where PN is the perpendicular and PG the normal at P ;
- (b) if $OM = y$, only one normal can be so drawn through O and, if OP be any other straight line drawn to the curve cutting the axis in K , $NK < NG$, as before;
- (c) if $OM < y$, two normals can be so drawn through O and, if OP be any other straight line drawn to the curve

greater or less than NG according as OP is or is not intermediate between the two normals (V. 51, 52).

The proofs are of course long and complicated. The length y is determined in this way:

(1) In the case of the parabola, measure MH towards the vertex equal to $\frac{1}{2}p$, and divide AH at N_1 so that $HN_1 = 2N_1A$. The length y is then taken such that

$$y : P_1N_1 = N_1H : HM,$$

where P_1N_1 is the ordinate passing through N_1 ;

(2) In the case of the hyperbola and ellipse, we have $AM > \frac{1}{2}p$, so that $CA : AM < AA' : p$; therefore, if H be taken on AM such that $CH : HM = AA' : p$, H will fall between A and M .

Take two mean proportionals CN_1, CI between CA and CH , and let P_1N_1 be the ordinate through N_1 .

The length y is then taken such that

$$y : P_1N_1 = (CM : MH) \cdot (HN_1 : N_1C).$$

In the case (b), where $OM = y$, O is the point of intersection of consecutive normals, i.e. O is the centre of curvature at the point P ; and, by considering the coordinates of O with reference to two coordinate axes, we can derive the Cartesian equations of the evolutes. E.g. (1) in the case of the parabola let the coordinate axes be the axis and the tangent at the vertex. Then $AM = x$, $OM = y$. Let $p = 4a$; then

$$HM = 2a, \quad N_1H = \frac{2}{3}(x - 2a), \quad \text{and} \quad AN_1 = \frac{1}{3}(x - 2a).$$

But $y^2 : P_1N_1^2 = N_1H^2 : HM^2$, by hypothesis,

$$\text{or} \quad y^2 : 4a \cdot AN_1 = N_1H^2 : 4a^2;$$

$$\begin{aligned} \text{therefore} \quad ay^2 &= AN_1 \cdot N_1H^2, \\ &= \frac{4}{27}(x - 2a)^3, \end{aligned}$$

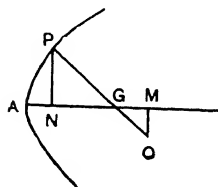
$$\text{or} \quad 27ay^2 = 4(x - 2a)^3.$$

(2) In the case of the hyperbola or ellipse we naturally take

ceding propositions.

Construction of normals.

The next section of the Book (V. 55-63) relates to the construction of normals through various points according to position within or without the conic and in relation to the axes. It is proved that one normal can be drawn through any internal point and through any external point which lies on the axis through the vertex A . In particular, if O is a point below the axis AA' of an ellipse, and OM is perpendicular to AA' , then, if $AM > AC$, one normal can always be drawn through O cutting the axis between A and C , but more than one such normal (V. 55-7). The points on the curve at which the straight lines through O are normals are determined as the intersections of the conic with a certain



rectangular hyperbola. The proof of Apollonius is equivalent to the following analytical method. Let AA' be the axis of a conic, PGO one of the normals which passes through the point O , PN the ordinate at P ; and OM be drawn perpendicular to the axis.

Take as axes of coordinates the axes in the central conic, or in the case of the parabola, the axis and the tangent at the vertex.

If then (x, y) be the coordinates of P and (x_1, y_1) those of O , we have

$$\frac{y}{-y_1} = \frac{NG}{x_1 - x - NG}.$$

Therefore (1) for the parabola

$$\frac{y}{-y_1} = \frac{\frac{1}{2}p}{x_1 - x - \frac{1}{2}p},$$

or

$$xy - (x_1 - \frac{1}{2}p)y - y_1 \cdot \frac{1}{2}p = 0;$$

(2) in the ellipse or hyperbola

$$xy \left(1 \mp \frac{b^2}{a^2}\right) - x_1y \pm \frac{b^2}{a^2} \cdot y_1x = 0.$$

The intersections of these rectangular hyperbolas re-

with the conics give the points at which the normals passing through O are normals.

Apollonius criticizes the use of the rectangular hyperbola in the use of the parabola as an unnecessary resort to a 'solid'; the meaning evidently is that the same points of intersection can be got by means of a certain circle taking the place of the rectangular hyperbola. We can, in fact, from equation (1) above combined with $y^2 = px$, obtain the

$$(x^2 + y^2) - (x_1 + \frac{1}{2}p)x - \frac{1}{2}y_1y = 0.$$

The Book concludes with other propositions about maxima and minima. In particular V. 68-71 compare the lengths of segments TQ , TQ' , where Q is nearer to the axis than Q' . V. 74 compare the lengths of two normals from a point from which only two can be drawn and the lengths of other right lines from O to the curve; V. 75-7 compare the lengths of three normals to an ellipse drawn from a point above the major axis, in relation to the lengths of other right lines from O to the curve.

Book VI is of much less interest. The first part (VI. 1-27) deals with equal (i.e. congruent) or similar conics and segments of conics; it is naturally preceded by some definitions including those of 'equal' and 'similar' as applied to conics and segments of conics. Conics are said to be similar if, the same number of ordinates being drawn to the axis at proportional distances from the vertices, all the ordinates are respectively proportional to the corresponding abscissae. The definition of equal segments is the same with diameter substituted for axis and with the additional condition that the angles between the base and diameter in each are equal. Two conics are equal if the ordinates to a diameter in each are equal to the respective diameters at equal angles and the corresponding parameters are equal; two ellipses or hyperbolas are equal if the ordinates to a diameter in each are equal to the respective diameters and the diameters are equally inclined to the respective diameters and the diameters

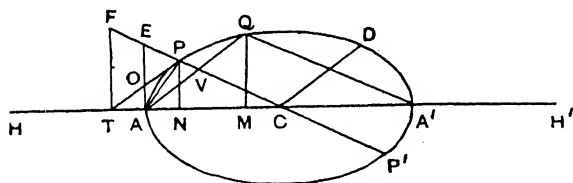
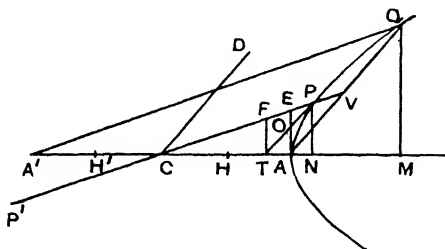
each make equal angles with them; all parabolas are similar (VI. 11, 12, 13). No conic of one of the three kinds (parabolas, hyperbolas or ellipses) can be equal or similar to a conic of either of the other two kinds (VI. 3, 14, 15). Let Qq , qq' be two segments of similar conics in which QQ' , qq' are the bases and PV , pv are the diameters bisecting them; then if PT , pt be the tangents at P , p and meet the axes at T , t making equal angles, and if $PV:PT = pv:pt$, the segments are similar and similarly situated, and conversely (VI. 17, 18). If ordinates be drawn to the axes of two parabolas, or the major and conjugate axes of two similar central conics, as PN , $P'N'$ and pn , $p'n'$ respectively, such that the ratios $AN:an$ and $A'N':a'n'$ are each equal to the ratio of the respective *latera recta*, the segments PP' , pp' will be similar; also PP' will not be similar to any segment in the other conic cut off by two ordinates other than pn , $p'n'$, and conversely (VI. 21, 22). If any conic be cut by two parallel planes making hyperbolic or elliptic sections, the sections will be similar but not equal (VI. 26).

The remainder of the Book consists of problems of construction; we are shown how in a given right cone to find a parabolic, hyperbolic or elliptic section equal to a given parabola, hyperbola or ellipse, subject in the case of the hyperbola to a certain *διορισμός* or condition of possibility (VI. 28-30); also how to find a right cone similar to a given cone and containing a given parabola, hyperbola or ellipse as a section of it, subject again in the case of the hyperbola to a certain *διορισμός* (VI. 31-3). These problems recall somewhat similar problems in I. 51-9.

Book VII begins with three propositions giving expressions for $AP^2 (= AN^2 + PN^2)$ in the same form as those for PN^2 in the statement of the ordinary property. In the parabola PN is measured along the axis produced (i.e. in the opposite direction to AN) and of length equal to the *latus rectum*, and it is proved that, for any point P , $AP^2 = AN \cdot NH$ (VII. 1). In the case of the central conics AA' is divided at H , internally for the hyperbola and externally for the ellipse (AH being

same is true if AA' is the minor axis of an ellipse and p corresponding parameter (VII. 2, 3).

AA' be divided at H' as well as H (internally for the parabola and externally for the ellipse) so that H is adjacent to A and H' to A' , and if $A'H:AH = AH':A'H' = AA':p$, then $AH, A'H'$ (corresponding to p in the proportion) are called by Apollonius *homologues*, and he makes considerable



use of the auxiliary points H, H' in later propositions from VII. 6 onwards. Meantime he proves two more propositions, VII. 4, 5, like VII. 1-3, are by way of lemmas. First, if CD be a semi-diameter parallel to the tangent at P to a central conic, and if the tangent meet the axis AA' in T , then

$$PT^2:CD^2 = NT:CN. \quad (\text{VII. 4.})$$

Draw AE, TF at right angles to CA to meet CP , and let AE and TF meet in O . Then, if p' be the parameter of the ordinates AE, TF , we have

$$\begin{aligned} \frac{1}{2} p' : PT &= OP : PE \\ &= PT : PF, \end{aligned} \quad (\text{I. 49, 50.})$$

$$\frac{1}{2} p' : PF = PT^2 : CD^2$$

Secondly, Apollonius proves that, if PN be a principal ordinate in a parabola, p the principal parameter, p' parameter of the ordinates to the diameter through P , $p' = p + 4AN$ (VII. 5); this is proved by means of the property as VII. 4, namely $\frac{1}{2}p':PT = OP:PE$.

Much use is made in the remainder of the Book of points Q and M , where AQ is drawn parallel to the conjugate diameter CD to meet the curve in Q , and M is the focus, the principal ordinate at Q ; since the diameter CP is parallel to both AA' and QA , it follows that $A'Q$ is parallel to AM . Many ratios between functions of PP' , DD' are expressed in terms of AM , $A'M$, MH , MH' , AH , $A'H$, &c. The first positions of the Book proper (VII. 6, 7) prove, for instance, that $PP'^2:DD'^2 = MH':MH$.

For $PT^2:CD^2 = NT:CN = AM:A'M$, by similar triangles.

Also $CP^2:PT^2 = A'Q^2:AQ^2$.

Therefore, *ex aequali*,

$$\begin{aligned} CP^2:CD^2 &= (AM:A'M) \times (A'Q^2:AQ^2) \\ &= (AM:A'M) \times (A'Q^2:A'M.MH') \\ &\quad \times (A'M.MH':AM.MH) \times (AM.MH:AM) \\ &= (AM:A'M) \times (AA':AH') \times (A'M:AM) \\ &\quad \times (MH':MH) \times (A'H:AA'), \text{ by aid of VII. 6, 7.} \end{aligned}$$

Therefore $PP'^2:DD'^2 = MH':MH$.

Next (VII. 8, 9, 10, 11) the following relations are proved, namely

- (1) $AA'^2:(PP' \pm DD')^2 = A'H.MH':\{MH' \pm \sqrt{(MH.MH')}\}$
- (2) $AA'^2:PP'.DD' = A'H:\sqrt{(MH.MH')}$,
- (3) $AA'^2:(PP'^2 \pm DD'^2) = A'H:MH \pm MH'$.

The steps by which these results are obtained are as follows.

First,

$$\begin{aligned} AA'^2:PP'^2 &= A'H:MH' \\ &= A'H.MH':MH'^2. \end{aligned}$$

(This is proved thus:

$$\begin{aligned}
 A'Q^2 : A'M \cdot MH' &= AA' : AH' & (\text{VII. 2, 3}) \\
 &= AA' : A'H \\
 &= A'M \cdot AA' : A'M \cdot A'H,
 \end{aligned}$$

at, alternately,

$$\begin{aligned}
 A'M \cdot AA' : A'Q^2 &= A'M \cdot A'H : A'M \cdot MH' \\
 &= A'H : MH'.
 \end{aligned}$$

ext, $PP'^2 : DD'^2 = MH' : MH$, as above, (β)

$$= MH'^2 : MH \cdot MH',$$

ce $PP' : DD' = MH' : \sqrt{(MH \cdot MH')}$, (γ)

$$PP'^2 : (PP' \pm DD')^2 = MH'^2 : \{MH' \pm \sqrt{(MH \cdot MH')}\}^2;$$

bove follows from this relation and (α) *ex aequali*;

follows from (α) and (γ) *ex aequali*, and (3) from (α) (β) .

e now obtain immediately the important proposition that $\pm DD'^2$ is constant, whatever be the position of P on an ellipse or hyperbola (the upper sign referring to the ellipse), is equal to $AA'^2 \pm BB'^2$ (VII. 12, 13, 29, 30).

r $AA'^2 : BB'^2 = AA' : p = A'H : AH = A'H : A'H'$,

by construction;

fore $AA'^2 : AA'^2 \pm BB'^2 = A'H : HH'$;

from (α) above,

$$AA'^2 : PP'^2 = A'H : MH';$$

by means of (β) ,

$$\begin{aligned}
 PP'^2 : (PP' \pm DD')^2 &= MH' : MH' \pm MH \\
 &= MH' : HH'.
 \end{aligned}$$

ex aequali, from the last two relations, we have

A number of other ratios are expressed in terms of straight lines terminating at A, A', H, H', M, M' as follows (VII. 14–20).

In the ellipse $AA'^2:PP'^2 \sim DD'^2 = A'H:2CM$,
and in the hyperbola or ellipse (if p be the parameter of ordinates to PP')

$$AA'^2:p^2 = A'H.MH':MH^2,$$

$$AA'^2:(PP' \pm p)^2 = A'H.MH':(MH \pm MH')^2,$$

$$AA'^2:PP'.p = A'H:MH,$$

and $AA'^2:(PP'^2 \pm p^2) = A'H.MH':(MH'^2 \pm MH^2).$

Apollonius is now in a position, by means of all the relations, resting on the use of the auxiliary points H, H' , to compare different functions of any conjugate diameter with the same functions of the axes, and to show how they former vary (by way of increase or diminution) as P moves away from A . The following is a list of the functions compared, where for brevity I shall use a, b to represent AA', Bb ; a', b' to represent PP', DD' ; and p, p' to represent the parameters of the ordinates to AA', PP' respectively.

In a hyperbola, according as $a >$ or $< b, a' >$ or $< b'$, and the ratio $a':b'$ decreases or increases as P moves from A either side; also, if $a = b, a' = b'$ (VII. 21–3); in an ellipse $a:b > a':b'$, and the latter ratio diminishes as P moves from A to B (VII. 24).

In a hyperbola or ellipse $a+b < a'+b'$, and $a'+b'$ in the hyperbola increases continually as P moves farther from A , but in the ellipse increases till a', b' take the position of the equal conjugate diameters when it is a *maximum* (VII. 25, 26).

In a hyperbola in which a, b are unequal, or in an ellipse $a \sim b > a' \sim b'$, and $a' \sim b'$ diminishes as P moves away from A in the hyperbola continually, and in the ellipse till a', b' are the equal conjugate diameters (VII. 27).

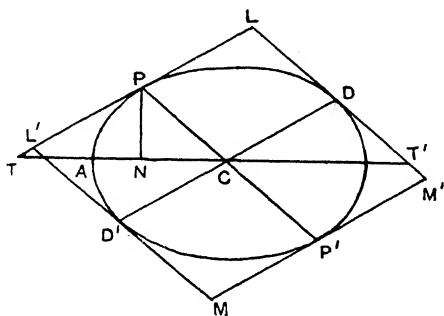
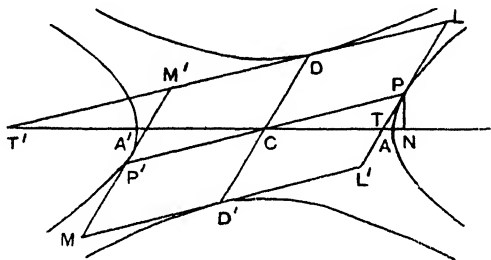
conjugate diameters in an ellipse or conjugate hyperbolas, and
 the tangents at their extremities form the parallelogram
 MM' , then

the parallelogram $LL'MM' = \text{rect. } AA'.BB'$.

The proof is interesting. Let the tangents at P, D respec-
 tively meet the major or transverse axis in T, T' .

Now (by VII. 4) $PT^2:CD^2 = NT:CN$;

Therefore $2\Delta CPT:2\Delta T'DC = NT:CN$.



$$2\Delta CPT:(CL) = PT:CD,$$

$$= CP:DT', \text{ by similar triangles,}$$

$$= (CL):2\Delta T'DC.$$

$$= PN \cdot \frac{CA}{CB} : CN \quad (I).$$

$$= PN \cdot CT : CT \cdot CN \cdot \frac{CB}{CA}$$

$$= 2 \Delta CPT : CA \cdot CB;$$

therefore

$$(CL) = CA \cdot CB.$$

The remaining propositions of the Book trace the various functions of the conjugate diameters, distinguishing the maximum values, &c. The functions treated are the following:

p' , the parameter of the ordinates to PP' in the hyperbola, according as AA' is (1) not less than p , the parameter corresponding to AA' , (2) less than p but not less than $\frac{1}{2}p$, or (3) less than $\frac{1}{2}p$ (VII. 33-5).

$PP' \sim p'$, as compared with $AA' \sim p$ in the hyperbola (VII. 33) or the ellipse (VII. 37).

$PP' + p'$ " " $AA' + p$ in the hyperbola (VII. 38-40) or the ellipse (VII. 39).

$PP' \cdot p'$ " " $AA' \cdot p$ in the hyperbola (VII. 41) or the ellipse (VII. 43).

$PP'^2 + p'^2$ " " $AA'^2 + p^2$ in the hyperbola, according as (1) AA' is not less than p , or (2) $AA' < p$, but $AA' > \frac{1}{2}p$, or (3) less than $\frac{1}{2}(AA' \sim p)^2$, or (4) $AA'^2 < \frac{1}{2}(AA' \sim p)^2$ (VII. 45-48).

$PP'^2 + p'^2$ " " $AA'^2 + p^2$ in the ellipse, according as AA'^2 is not greater than p^2 , or greater, than $(AA' + p)^2$ (VII. 47, 48).

$PP'^2 \sim p'^2$ " " $AA'^2 \sim p^2$ in the hyperbola, according as $AA' > p$ or $AA' < p$ (VII. 49, 50).

$PP'^2 \sim p'^2$ " " $AA'^2 \sim p^2$ or $BB'^2 \sim p_b^2$ in the ellipse, according as $PP' > p$ or $PP' < p$ (VII. 51).

contents can only be conjectured from Apollonius's own remark that it contained determinate conic problems for which Book VII was useful, particularly in determining limits of possibility. Unfortunately, the lemmas of Pappus do not enable us to form any clearer idea. But it is probable enough that the Book contained a number of problems having for their object the finding of conjugate diameters in a given conic such that certain functions of their lengths have given values. It was on this assumption that Halley attempted a restoration of the Book.

If it be thought that the above account of the *Conics* is disproportionately long for a work of this kind, it must be remembered that the treatise is a great classic which deserves to be more known than it is. What militates against its being read in its original form is the great extent of the exposition (it contains 387 separate propositions), due partly to the Greek habit of proving particular cases of a general proposition separately from the proposition itself, but more to the cumbrousness of the enunciations of complicated propositions in general terms (without the help of letters to denote particular points) and to the elaborateness of the Euclidean form, to which Apollonius adheres throughout.

Other works by Apollonius.

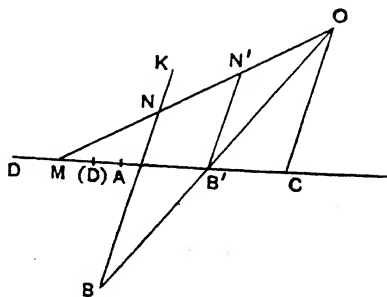
Pappus mentions and gives a short indication of the contents of six other works of Apollonius which formed part of the *Treasury of Analysis*.¹ Three of these should be mentioned in close connexion with the *Conics*.

- (a) *On the Cutting-off of a Ratio* (λόγου ἀποτομή),
two Books.

This work alone of the six mentioned has survived, and that only in the Arabic; it was published in a Latin translation by Edmund Halley in 1706. It deals with the general problem, 'Given two straight lines, parallel to one another or intersecting, and a fixed point on each line, to draw through

¹ Pappus, vii, pp. 640–8, 660–72.

each line (measured from the fixed points) bearing a ratio to one another.' Thus, let A, B be fixed points on two given straight lines AC, BK , and let O be the point. It is required to draw through O a straight line cutting the given straight lines in points M, N respectively.



such that AM is to BN in a given ratio. The two Books of the treatise discussed the various possible cases of this problem which arise according to the relative positions of the given straight lines and points, and also the necessary conditions and limits of possibility in cases where a solution is always possible. The first Book begins by supposing the given lines to be parallel, and discusses the different cases which arise; Apollonius then passes to the cases in which the straight lines intersect, but one of the given points, A or B , is at the intersection of the two lines. Book II proceeds to the general case shown in the above figure, and first proves that the general case can be reduced to the case in Book I where one of the given points, A or B , is at the intersection of the two lines. The reduction is easy. For join OB meeting AC in B' , and draw $B'N'$ parallel to BN to meet OM in N' . Then the ratio $B'N':BN$, being equal to the ratio $OB':OB$, is constant. Since, therefore, $BN:AM$ is a given ratio, the ratio $B'N':AM$ is also given.

Apollonius proceeds in all cases by the orthodox method of analysis and synthesis. Suppose the problem solved and a line ONM drawn through O in such a way that $B'N':AM$ is a given ratio $= \lambda$, say.

AM such that $OC:AD = \lambda = B'N':AM$.

$$\begin{aligned} \text{en} \quad AM:AD &= B'N':OC \\ &= B'M:CM; \end{aligned}$$

$$\text{fore} \quad MD:AD = B'C:CM,$$

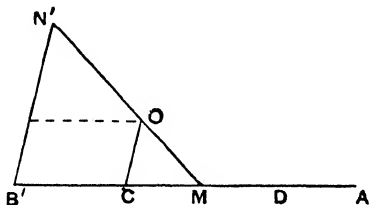
$CM.MD = AD.B'C$, a given rectangle.

nce the problem is reduced to one of *applying to CD a angle $(CM.MD)$ equal to a given rectangle $(AD.B'C)$ but g short by a square figure*. In the case as drawn, what- be the value of λ , the solution is always possible because iven rectangle $AD.CB'$ is always less than $CA.AD$, and fore always less than $\frac{1}{4}CD^2$; one of the positions of ls between A and D because $CM.MD < CA.AD$.

e proposition III. 41 of the *Conics* about the intercepts on two tangents to a parabola by a third tangent 55-6 above) suggests an obvious application of our pro- We had, with the notation of that proposition,

$$Pr:rq = rQ:Qp = qp:pr.$$

ose that the two tangents qP, qR are given as fixed nts with their points of contact P, R . Then we can another tangent if we can draw a straight line ecting qP, qR in such a way that $Pr:rq = qp:pr$ or $r = qR:pR$, i.e. $qr:pR = Pq:qR$ (a constant ratio); e have to draw a straight line such that the intercept y qP measured from q has a given ratio to the intercept on qR measured from R . This is a particular case of problem to which, as a matter of fact, Apollonius devotes al attention. In the annexed figure the letters have the



meaning as before, and $N'M$ has to be drawn through O that $B'N':AM = \lambda$. In this case there are limits to

Apollonius begins by stating the limiting case, saying that he obtains a solution in a special manner in the case where I is the middle point of CD , so that the rectangle $CM.MD$ is a maximum. $CB'.AD$ has its maximum value.

The corresponding limiting value of λ is determined by finding the corresponding position of D or M .

We have $B'C:MD = CM:AD$, as before,
 $= B'M:MA$;

whence, since $MD = CM$,

$$B'C:B'M = CM:MA \\ = B'M:B'A,$$

so that

$$B'M^2 = B'C.B'A.$$

Thus M is found and therefore D also.

According, therefore, as λ is less or greater than the particular value of $OC:AD$ thus determined, Apollonius finds one solution or two solutions.

Further, we have

$$AD = B'A + B'C - (B'D + B'C) \\ = B'A + B'C - 2B'M \\ = B'A + B'C - 2\sqrt{B'A.B'C}.$$

If then we refer the various points to a system of co-ordinates in which $B'A, B'N'$ are the axes of x and y , and we denote O by (x, y) and the length $B'A$ by h ,

$$\lambda = OC/AD = y/(h+x-2\sqrt{hx}).$$

If we suppose Apollonius to have used these results for the parabola, he cannot have failed to observe that the limiting case described is that in which O is on the parabola, while $N'OM$ is the tangent at O ; for, as above,

$B'M:B'A = B'C:B'M = N'O:N'M$, by parallels,
 so that $B'A, N'M$ are divided at M, O respectively in the same proportion.

fixed tangents, then if h, k be those lengths,

$$\frac{k}{h} = \frac{y}{h+x-2\sqrt{hx}},$$

which can easily be reduced to

$$\left(\frac{x}{h}\right)^{\frac{1}{2}} + \left(\frac{y}{k}\right)^{\frac{1}{2}} = 1,$$

equation of the parabola referred to the two fixed tangents as axes.

(β) *On the cutting-off of an area* (χωρίου ἀποτομή), two Books.

This work, also in two Books, dealt with a similar problem, the difference that the intercepts on the given straight line measured from the given points are required, not to be in a given ratio, but to contain a given rectangle. Halley made an attempted restoration of this work in his edition of *De sectione rationis*.

The general case can here again be reduced to the more particular one in which one of the fixed points is at the intersection of the two given straight lines. Using the same construction as before, but with D taking the position shown by (D) in the figure, we take that point such that

$$OC \cdot AD = \text{the given rectangle.}$$

We have then to draw $ON'M$ through O such that

$$B'N' \cdot AM = OC \cdot AD,$$

$$B'N' : OC = AD : AM.$$

But, by parallels, $B'N' : OC = B'M : CM$;

$$\begin{aligned} \text{therefore} \quad AM : CM &= AD : B'M \\ &= MD : B'C, \end{aligned}$$

$$\text{that} \quad B'M \cdot MD = AD \cdot B'C.$$

Hence, as before, the problem is reduced to an application of the rectangle in the well-known manner. The complete

If the two straight lines are parallel, the solution of the problem gives a means of drawing any number of tangents to an ellipse when two parallel tangents, their points of contact, and the length of the parallel semi-diameter are given (see *Conics*, III. 42). In the case of the hyperbola the intercepts made by any tangent on the asymptotes are equal to a constant rectangle. Accordingly¹ the drawing of the tangents depends upon the particular case of our problem in which the fixed points are the intersection of the two fixed lines.

(γ) *On determinate section (διορισμένη τομή)*, two

The general problem here is, Given four points A, B, C, D and a straight line, to determine another point P on the straight line such that the ratio $AP \cdot CP : BP \cdot DP$ is a given value. It is clear from Pappus's account¹ of the contents of this work, and from his extensive collection of lemmata, that the different propositions in it, that the question is exhaustively discussed. To determine P by means of an equation

$$AP \cdot CP = \lambda \cdot BP \cdot DP,$$

where A, B, C, D, λ are given, is in itself an easy matter, and the problem can at once be put into the form of a quadratic equation, and the Greeks would have no difficulty in applying it to the usual *application of areas*. If, however (as we may fairly suppose), it was intended for application in the investigations, the complete discussion of it would include not only the finding of a solution, but also the determination of the limits of possibility and the number of solutions for different positions of the point-pairs A, B, D , for the cases in which the points in either pair are infinitely distant, or in which one of the points is infinitely distant, and so on. This agrees with what we find in Pappus, who makes a *series* of point-pairs determined by the equation for different values of λ , yet the treatise contained what amounts to

¹ Pappus, vii, pp. 642-4.

Theory of Involution. Pappus says that the separate problems were dealt with in which the given ratio was that of (1) the square of one abscissa measured from the required point or (2) the rectangle contained by two such abscissae to any one of the following: (1) the square of one abscissa, (2) the rectangle contained by one abscissa and another separate line of given length independent of the position of the required point, (3) the rectangle contained by two abscissae. We learn also that maxima and minima were investigated. From the lemmas, too, we may draw other conclusions, e. g.

that, in the case where $\lambda = 1$, or $AP \cdot CP = BP \cdot DP$, Apollonius used the relation $BP : DP = AB \cdot BC : AD \cdot DC$,

that Apollonius probably obtained a double point E of the section determined by the point-pairs A, C and B, D by means of the relation

$$AB \cdot BC : AD \cdot DC = BE^2 : DE^2.$$

A possible application of the problem was the determination of the points of intersection of the given straight line with a circle determined as a four-line locus, since A, B, C, D are in the points of intersection of the given straight line with four lines to which the locus has reference.

(δ) *On Contacts or Tangencies* (ἐπαφαί), two Books.

Pappus again comprehends in one enunciation the varieties of problems dealt with in the treatise, which we may reproduce as follows: *Given three things, each of which may be a point, a straight line or a circle, to draw a circle which shall pass through each of the given points (so far as it is possible that are given) and touch the straight lines or circles.*¹ The possibilities as regards the different data are three. We may have any one of the following: (1) three

first two are treated in Eucl. IV; Book I of the treatise treated of (3), (4), (5), (6), (8), (9), while (7), (10), that of two straight lines and a circle, and (10), that of two circles, occupied the whole of Book II.

The last problem (10), where the data are three circles, has exercised the ingenuity of many distinguished geometers, including Vieta and Newton. Vieta (1540-1603) reduced the problem to Adrianus Romanus (van Roomen, 1561-1615) solved it by means of a hyperbola. Vieta was not satisfied with this, and rejoined with his *Apollonius Gallice* (1585) in which he solved the problem by plane methods. A similar problem of the same kind is given by Newton in his *Arithmetica Universalis* (Prob. xlvii), while an equivalent problem is solved by means of two hyperbolas in the *Principia* (Book I, xvi). The problem is quite capable of a 'plane' solution, as a matter of fact, it is not difficult to restore the original solution of Apollonius (which of course used the 'plane' methods depending on the straight line and circle only), by using the lemmas given by Pappus. Three things are required for the solution. (1) A proposition, used by Pappus and easily proved, that, if two circles touch internally, any straight line through the point of contact divides the circles into segments respectively similar. (2) A proposition that, given three circles, their six centres lie three by three on three straight lines. This proposition, though not proved in Pappus, is certainly known to the ancient geometers; it is evident that Pappus omitted to prove it because it was actually proved by Apollonius in his treatise. (3) An auxiliary proposition given by Pappus and enunciated by him as follows.² Given a triangle ABC , and given three points D, E, F in a straight line, to draw a line that will inflect (the broken line) DAE (to the circle) so as to be tangent to the circle in a straight line with CF ; in other words, to inscribe in the circle a triangle the sides of which, when produced, pass respectively through three given points lying in a straight line. This problem is interesting as a typical example of ancient analysis followed by synthesis. Suppose that

¹ Pappus, iv, pp. 194-6.

² *Ib.*

i.e. suppose DA, EA drawn to the circle cutting it in B, C such that BC produced passes through F .

Now BG parallel to DF ; join GC
Produce it to meet DE in H .

$$\angle BAC = \angle BGC$$

$$= \angle CHF$$

$$= \text{supplement of } \angle CHD;$$

therefore A, D, H, C lie on a circle, and

$$DE \cdot EH = AE \cdot EC.$$

For $AE \cdot EC$ is given, being equal to the square on the tangent from E to the circle; and DE is given; therefore HE is given, and therefore the point H .

F is also given; therefore the problem is reduced to finding HC, FC to meet the circle in such a way that, if BC produced meet the circle again in G, B , the straight line BG is parallel to HF : a problem which Pappus has justly solved.¹

Suppose this done, and draw BK the tangent at B meeting DE in K . Then

$$\angle KBC = \angle BGC, \text{ in the alternate segment,}$$

$$= \angle CHF.$$

Also the angle CFK is common to the two triangles KBF, FCH , therefore the triangles are similar, and

$$CF:FH = KF:FB,$$

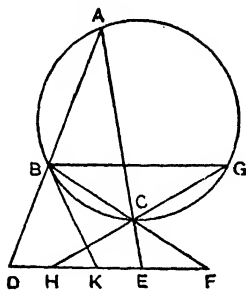
$$HF \cdot FK = BF \cdot FC.$$

Now $BF \cdot FC$ is given, and so is HF ;

therefore FK is given, and therefore K is given.

The synthesis is as follows. Take a point H on DE such that $DE \cdot EH$ is equal to the square on the tangent from E to the circle.

Then take K on HF such that $HF \cdot FK =$ the square on the



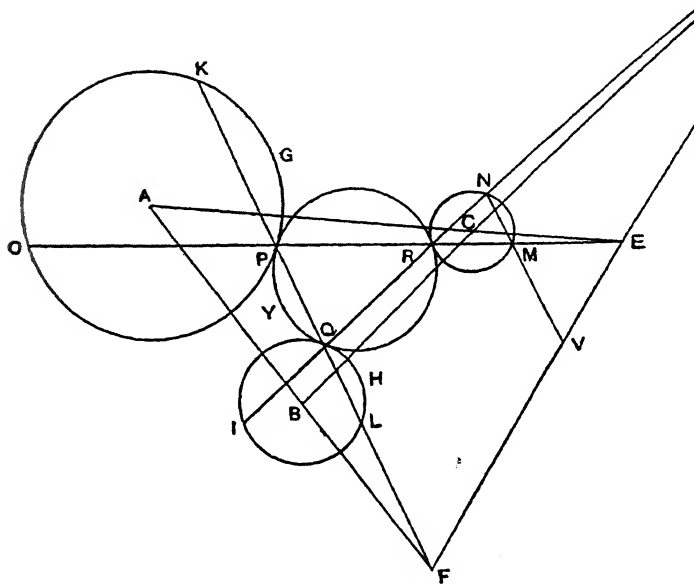
that BG is parallel to DF .

Now join EC , and produce it to meet the circle again at H , and join AB .

We have only to prove that AB, BD are in one straight line.

Since $DE \cdot EH = AE \cdot EC$, the points A, D, H, C are cyclic.

Now the angle CHF , which is the supplement of the angle



CHD , is equal to the angle BGC , and therefore angle BAC .

Therefore the angle BAC is equal to the supplement of the angle DHC , so that the angle BAC is equal to the angle BDC , and AB, BD are in a straight line.

The problem of Apollonius is now easy. We will consider the case in which the required circle touches all the three circles externally as shown in the figure. Let the radii

circles be a, b, c and their centres A, B, C . Let D, E, F be the external centres of similitude so that $BD:DC=b:c$, &c. Suppose the problem solved, and let P, Q, R be the points of contact. Let PQ produced meet the circles with centres A, B again in K, L . Then, by the proposition (1) above, the triangles KGP, QHL are both similar to the segment PYQ ; therefore they are similar to one another. It follows that PQ produced beyond L passes through F . Similarly QR, PR produced pass respectively through D, E .

PE, QD meet the circle with centre C again in M, N . Since the segments PQR, RNM being similar, the angles PRM, RNM are equal, and therefore MN is parallel to PQ . Hence NM to meet EF in V .

Then $EV:EF=EM:EP=EC:EA=c:a$;

therefore the point V is given.

Accordingly the problem reduces itself to this: Given three circles V, E, D in a straight line, it is required to draw DR, ER from a point R on the circle with centre C so that, if DR, ER meet the circles again in N, M, NM produced shall pass through V . This is the problem of Pappus just solved.

As R is found, and DR, ER produced meet the circles with centres B and A in the other required points Q, P respectively.

(ε) *Plane loci*, two Books.

Pappus gives a pretty full account of the contents of this book, which has sufficed to enable restorations of it to be made by three distinguished geometers, Fermat, van Schooten, and (most completely) by Robert Simson. Pappus begins his account by a classification of loci on two different plans. Under the first classification loci are of three kinds: (1) *ἐφεκτικοί*, *holding-in* or *fixed*; in this case the locus of a point is a point, of a line a line, and of a solid a surface, where presumably the line or solid can only move on a fixed surface so that it does not change its position; (2) *ἀνεκ-*

plane, solid, and linear loci, plane loci being straight lines and circles only, solid loci conic sections only, and linear loci those which are not straight lines nor circles nor conic sections. The loci dealt with in our treatise are only straight lines or circles. The proof of the propositions is of course enormously facilitated by the use of Cartesian coordinates, and many of the loci are geometrical equivalents of fundamental theorems in algebraic geometry. Pappus begins with a general enunciation, including a number of propositions, in which terms, which, though apparently confused, are not difficult to follow out:

‘If two straight lines be drawn, from one given point to two, which are (α) in a straight line or (β) parallel, or (γ) include a given angle, and either (α) bear a given ratio one another or (β) contain a given rectangle, then, if the extremity of one of the lines is a plane locus in position, the locus of the extremity of the other will be a plane locus given in position, which will sometimes be of the same kind as the former, sometimes of the other kind. It will sometimes be similarly situated with reference to the straight line, and sometimes contrarily, according to particular differences in the suppositions.’

(The words ‘with reference to the straight line’ are of course with reference to the straight line which is the locus of the extremity of the other line. The straight line is presumably some obvious straight line in each figure, e.g., when there are two given points, the straight line joining them.) After quoting three obvious loci by Charmandrus, Pappus gives three loci which, though containing an unnecessary restriction in the third case, are due to the statement that any equation of the first degree in the coordinates inclined at fixed angles to (α) two axes, one perpendicular or oblique, (β) to any number of axes, represents a straight line. The enunciations (5-7) are as follows:

5. ‘If, when a straight line is given in magnitude and position, it is moved so as always to be parallel to a certain straight line given in position, one of the extremities (of the moving straight line), lies on a straight line given in position.’

¹ Pappus, vii, pp. 660. 18-662. 5.

² *Ib.* vii, pp. 662. 6-664. 1.

³ *Ib.*, pp. 664. 20-666. 6.

other extremity will also lie on a straight line given in position.'

(That is, $x = a$ or $y = b$ in Cartesian coordinates represents a straight line.)

6. 'If from any point straight lines be drawn to meet at given angles two straight lines either parallel or intersecting, and if the straight lines so drawn have a given ratio to one another or if the sum of one of them and a line to which the other has a given ratio be given (in length), then the point will lie on a straight line given in position.'

(This includes the equivalent of saying that, if x, y be the coordinates of the point, each of the equations $x = my$, $x + my = c$ represents a straight line.)

7. 'If any number of straight lines be given in position, and straight lines be drawn from a point to meet them at given angles, and if the straight lines so drawn be such that the rectangle contained by one of them and a given straight line added to the rectangle contained by another of them and (another) given straight line is equal to the rectangle contained by a third and a (third) given straight line, and similarly with the others, the point will lie on a straight line given in position.'

(Here we have trilinear or multilinear coordinates proportional to the distances of the variable point from each of the three or more fixed lines. When there are three fixed lines, the statement is that $ax + by = cz$ represents a straight line. The precise meaning of the words 'and similarly with the the others' or 'of the others'—*καὶ τῶν λοιπῶν ὁμοίως*—is uncertain; the words seem to imply that, when there were more than three rectangles $ax, by, cz \dots$, two of them were taken to be equal to the sum of all the others; but it is quite possible that Pappus meant that any linear equation between these rectangles represented a straight line. Precisely how far Apollonius went in generality we are not in a position to judge.)

(b) they contain a given rectangle or (c) the sum or difference of figures of given species described on them respectively equal to a given area, the point will lie on a straight line given in position.¹

The contents of Book II are equally interesting. Some of the enunciations shall for brevity be given by means of symbols instead of in general terms. If from two given points two straight lines be 'inflected' ($\kappa\lambda\alpha\sigma\theta\acute{\omega}\sigma\iota\nu$) to a point (1), if $AP^2 \sim BP^2$ is given, the locus of P is a straight line; (2) if AP, BP are in a given ratio, the locus is a straight line or a circle [this is the proposition quoted by Eutocius in his commentary on the *Conics*, but already known to Aristotle]; (3) if AP^2 is 'greater by a given area than in a given ratio to BP^2 ', i.e. if $AP^2 = a^2 + m \cdot BP^2$, the locus is a circle given in position. An interesting proposition is (5) that, 'If from a number of given points whatever straight lines be inflected to one point, and the figures (given in species) described on them be together equal to a given area, the point will lie on a circumference (circle) given in position'; that is to say, if $\alpha \cdot AP^2 + \beta \cdot BP^2 + \gamma \cdot CP^2 + \dots =$ a given area (where $\alpha, \beta, \gamma, \dots$ are constants), the locus of P is a circle. (3) states that if AN be a fixed straight line and A a fixed point on it, and if AP be any straight line drawn to a point P such that AP is perpendicular to AN , $AP^2 = a \cdot AN$ or $a \cdot BN$, where a is a given length and B is another fixed point on AN , the locus of P is a circle given in position; this is equivalent to the fact that, if A be the origin, AN the axis of x , $x = AN, y = PN$ be the coordinates of P , the locus $x^2 + y^2 = a(x - b)$ is a circle. (6) is somewhat obscurely enunciated: 'If from two given points straight lines be inflected (to a point), and from the point (of concurrence) a straight line be drawn parallel to a straight line given in position, cutting off from another straight line given in position a certain intercept measured from a given point on it, and if the figures (given in species) described on the two inflected lines be equal to the rectangle contained by a given straight line and the intercept, the point at which the straight line

¹ Pappus, vii, p. 666. 7-13.

sted lies on a circle given in position.' The meaning is to be this: Given two fixed points A, B , a length a , a straight line OX with a point O fixed upon it, and a direction represented, say, by any straight line OZ through O , then, AP, BP be drawn to P , and PM parallel to OZ meets OX the locus of P will be a circle given in position if

$$\alpha \cdot AP^2 + \beta \cdot BP^2 = a \cdot OM,$$

where α, β are constants. The last two loci are again more fully expressed, but the sense is this: (7) If PQ be any chord of a circle passing through a fixed internal point O , and R an external point on PQ produced such that either $OR^2 = PR \cdot RQ$ or (b) $OR^2 + PO \cdot OQ = PR \cdot RQ$, the locus of R is a straight line given in position. (8) is the reciprocal of (7): Given the fixed point O , the straight line which is the locus of R , and also the relation (a) or (b), the locus of P is a circle.

(9) *Νεύσεις* (*Vergings* or *Inclinations*), two Books.

As we have seen, the problem in a *νέυσις* is to place between two straight lines, a straight line and a curve, or two curves, a straight line of given length in such a way that it *verges* towards a fixed point, i.e. it will, if produced, pass through a fixed point. Pappus observes that, as we come to particular cases, the problem will be 'plane', 'solid' or 'linear', according to the nature of the particular hypotheses; but a selection had been made from the class which could be solved by plane methods, i.e. by means of the straight line and circle, the object being to give those which were more generally useful in geometry. The cases which were the cases thus selected and proved.¹

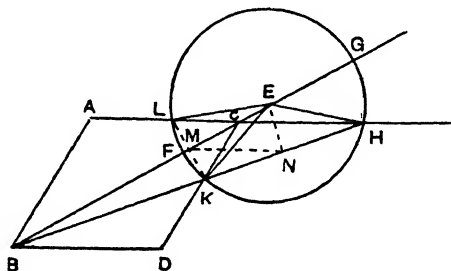
Given (a) a semicircle and a straight line at right angles to its base, or (b) two semicircles with their bases in a straight line, to insert a straight line of given length verging to an angle of the semicircle [or of one of the semicircles].

III. Given a circle, to insert a chord of given length verging to a given point.

In Book I of Apollonius's work there were four cases I (a), two cases of III, and two of II; the second Book contained ten cases of I (b).

Restorations were attempted by Marino Ghetaldi (*Apollonius redivivus*, Venice, 1607, and *Apollonius redivivus . . . Liber secundus*, Venice, 1613), Alexander Anderson (in a *Supplementum Apollonii redivivi*, 1612), and Samuel Horsley (Oxford, 1770); the last is much the most complete.

In the case of the rhombus (II) the construction of Apollonius can be restored with certainty. It depends on a lemma given by Pappus, which is as follows: Given a rhombus AD with diagonal BC produced to E , if F be taken on BC such that BF is a mean proportional between BE and EC , and if a circle



described with E as centre and EF as radius cutting AC in L and AC produced in H , then shall B, K, H be in a straight line.¹

Let the circle cut AC in L , join LK meeting BC in M , and join HE, LE, KE .

Since now CL, CK are equally inclined to the diameter CE of the circle, $CL = CK$. Also $EL = EK$, and it follows that the triangles ECK, ECL are equal in all respects, so that

$$\angle CKE = \angle CLE = \angle CHE.$$

Therefore the triangles BEK , KEC , which have the angle BEK common, are similar, and

$$\angle CBK = \angle CKE = \angle CHE \text{ (from above).}$$

But $\angle HCE = \angle ACB = \angle BCK$.

Therefore in the triangles CBK , CHE two angles are respectively equal, so that $\angle CEH = \angle CKB$ also.

But since $\angle CKE = \angle CHE$ (from above), K , C , E , H are concyclic.

Hence $\angle CEH + \angle CKH =$ (two right angles);
therefore, since $\angle CEH = \angle CKB$,

$$\angle CKB + \angle CKH = \text{(two right angles),}$$

and BKH is a straight line.

It is certain, from the nature of this lemma, that Apollonius made his construction by drawing the circle shown in the figure.

He would no doubt arrive at it by analysis somewhat as follows.

Suppose the problem solved, and HK inserted as required ($= k$).

Bisect HK in N , and draw NE at right angles to KH meeting BC produced in E . Draw KM perpendicular to BC , and produce it to meet AC in L . Then, by the property of the rhombus, $LM = MK$, and, since $KN = NH$ also, MN is parallel to LH .

Now, since the angles at M , N are right, M , K , N , E are concyclic.

Therefore $\angle CEK = \angle MNK = \angle CHK$, so that C , K , H , E are concyclic.

Therefore $\angle BCD =$ supplement of $KCE = \angle EHK = \angle EKH$, and the triangles EKH , DCB are similar.

Lastly,

$$\angle ERK = \angle EKH - \angle CEK = \angle EHK - \angle CHK - \angle EHC - \angle EKC.$$

$$EK:KH = DC:CB,$$

and, since the ratio $DC:CB$, as well as KH , is given, is given.

The construction then is as follows.

If k be the given length, take a straight line p such

$$p:k = AB:BC;$$

apply to BC a rectangle $BE \cdot EC$ equal to p^2 and exceed a square; then with E as centre and radius equal to p describe a circle cutting AC produced in H and CD in K . HK equal to k and, by Pappus's lemma, verges towards B .

Pappus adds an interesting solution of the same problem with reference to a square instead of a rhombus; the solution is by one Heraclitus and depends on a lemma which he also gives.¹

We hear of yet other lost works by Apollonius.

(η) *A Comparison of the dodecahedron with the icosahedron*. This is mentioned by Hypsicles in the preface to the second edition of Book XIV of Euclid. Like the *Conics*, it appeared in two editions, the second of which contained the proposition that if there be a dodecahedron and an icosahedron inscribed in one and the same sphere, the surfaces of the solids are in the same ratio as their volumes; this was established by showing that the perpendiculars from the centre of the sphere to a pentagonal face of the dodecahedron and to a triangular face of the icosahedron are equal.

(θ) Marinus on Euclid's *Data* speaks of a *General* (καθόλου πραγματεία) in which Apollonius used the term *assigned* (τεταγμένον) as a comprehensive term to describe a *datum* in general. It would appear that this work must have dealt with the fundamental principles of mathematics, definitions, axioms, &c., and that to it must be referred the various remarks on such subjects attributed to Apollonius. Proclus, the elucidation of the notion of a line, the de-

¹ Pappus, vii, pp. 780-4.

ane and solid angles, and his attempts to prove the axioms ; must also have included the three definitions (13-15) in Euclid's *Data* which, according to a scholium, were due to Apollonius and must therefore have been interpolated (they are definitions of *κατηγμένη*, *ἀνηγμένη*, and the elliptical case *παρὰ θέσει*, which means 'parallel to a straight line in position'). Probably the same work also contained Apollonius's alternative constructions for the problems of I. 10, 11 and 23 given by Proclus. Pappus speaks in mention by Apollonius 'before his own elements' of the locus of locus called *ἑκτατικός*, and it may be that the treatise in question is referred to rather than the *Plane Loci*.

The work *On the Cochlias* was on the cylindrical helix. It included the theoretical generation of the curve on the surface of the cylinder, and the proof that the curve is *isoperimetric* or uniform, i.e. such that any part will fit upon any other.

A work on *Unordered Irrationals* is mentioned by Proclus, and a scholium on Eucl. X. 1 extracted from Pappus's commentary remarks that 'Euclid did not deal with all rationals and irrationals, but only with the simplest kinds by combination of which an infinite number of irrationals are formed, of which latter Apollonius also gave some'. A like effect is a passage of the fragment of Pappus's commentary on Eucl. X discovered in an Arabic translation by Woepcke: 'it was Apollonius who, besides the *ordered* irrational magnitudes, showed the existence of the *unordered*, and by accurate methods set forth a great number of them'. The hints given by the author of the commentary seem to imply that Apollonius's extensions of the theory of irrationals took two directions, (1) generalizing the *medial* straight line of Euclid, on the basis that, between two lines commensurable in square (only), we may take not only one sole medial line but

(λ) *On the burning-mirror* (περὶ τοῦ πυρίου) is the another work of Apollonius mentioned by the author of *Fragmentum mathematicum Bobiense*, which is attributed by Heiberg to Anthemius but is more likely (judging by the use of antiquated terminology) to belong to a much earlier date. The fragment shows that Apollonius discussed the spherical form of mirror among others. Moreover, the fragment by Anthemius himself (on burning mirrors) proves the property of mirrors of parabolic section, using the properties of the parabola (a) that the tangent at any point makes equal angles with the axis and with the focal distance of the focus and (b) that the distance of any point on the curve from the focus is equal to its distance from a certain straight line (our 'directrix'); and we can well believe that the parabolic form of mirror was also considered in Apollonius's work, although that he was fully aware of the focal properties of the parabola notwithstanding the omission from the *Conics* of all mention of the focus of a parabola.

(μ) In a work called ὀκυτόκιον ('quick-delivery') Apollonius is said to have found an approximation to the value of π by a different calculation (from that of Archimedes), bringing it within closer limits'.¹ Whatever these closer limits may have been, they were considered to be less suitable for practical use than those of Archimedes.

It is a moot question whether Apollonius's system of arithmetical notation (by tetrads) for expressing large numbers and performing the usual arithmetical operations with them, as described by Pappus, was included in this same work. Heiberg thinks it probable, but there does not seem to be any necessary reason why the notation for large numbers, classifying them into myriads, double myriads, triple myriads, i.e. according to powers of 10,000, need have been connected with the calculation of the value of π , unless indeed the numbers used in the calculation were so large as to require the tetradic system for the handling of them.

Astronomy.

are told by Ptolemaeus Chennus¹ that Apollonius was famous for his astronomy, and was called ϵ (Epsilon) because the form of that letter is associated with that of the moon, to which his accurate researches principally related. Hippolytus of Rome made the distance of the moon's circle from the surface of the earth to be 500 myriads of stades.² This figure hardly can be right, for, the diameter of the earth being, according to Eratosthenes's evaluation, about eight myriads of stades, this would make the distance of the moon from the earth about 125 times the earth's radius. This is an unlikely supposition, seeing that Aristarchus had given limits for the ratios between the distance of the moon and its diameter, and between the diameters of the moon and the earth, which lead out 19 as the ratio of the moon's distance to the earth's diameter. Tannery suggests that perhaps Hippolytus made a mistake in copying from his source and took the figure of 5000 stades to be the length of the radius instead of the diameter of the moon's orbit.

But we have better evidence of the achievements of Apollonius in astronomy. In Ptolemy's *Syntaxis*³ he appears as an authority upon the hypotheses of epicycles and eccentrics needed to account for the apparent motions of the planets. The propositions of Apollonius quoted by Ptolemy contain statements of the alternative hypotheses, and from this it was at one time concluded that Apollonius invented both hypotheses. This, however, is not the case. The hypothesis of epicycles was already involved, though with limited application, in the theory of Heraclides of Pontus. The two inferior planets, Mercury and Venus, revolve in circles like satellites round the sun, while the sun itself moves in a circle round the earth; that is, the two planets move on epicycles about the material sun as moving centre. In order to explain the motions of the superior planets by means of epicycles it was necessary to conceive of an epicycle

another hypothesis, that of eccentrics, was invented to account for the movements of the superior planets only. We are at an earlier stage when we come to Apollonius. His enunciation of the hypothesis is that he understood the theory of epicycles in all its generality, but he states specifically that the theory of eccentrics is to be applied to the three planets which can be at any distance from the sun. The reason why he says that the hypothesis will not serve for the inferior planets is that in order to make it serve, we should have to suppose the planet described by the centre of the eccentric circle to be farther from the sun than the eccentric circle itself. (Even this generalization was made later, at or before the time of Hipparchus.) Apollonius further says in his enunciation about the eccentric hypothesis that the centre of the eccentric circle moves about the centre of the zodiac in the direct order of the signs and *at a speed equal to that of the sun*, while the star moves on the eccentric circle about its centre in the inverse order of the signs and at a speed equal to the anomaly'. It is clear from this that the theory of eccentrics was invented for the specific purpose of accounting for the movements of Mars, Jupiter, and Saturn about the sun and for that purpose alone. This explanation, together with the use of epicycles about the sun as centre to account for the motions of Venus and Mercury, amounted to a new system of Tycho Brahe; that system was therefore anticipated by some one intermediate in date between Heraclides and Apollonius and probably nearer to the latter, or even by Apollonius himself who took this important step. If it was, then Apollonius, coming after Aristarchus of Samos, would be exactly the Tycho Brahe of the classical age of antiquity. The actual propositions quoted by Pappus proved by Apollonius among others show mathematically that at what points, under each of the two hypotheses, the planet in forward motion changes into apparent retrograde motion, or vice versa, or the planet appears to be *stationary*.

XV

THE SUCCESSORS OF THE GREAT GEOMETERS

WITH Archimedes and Apollonius Greek geometry reached its culminating point. There remained details to be filled in, and no doubt in a work such as, for instance, the *Conics* geometers of the requisite calibre could have found propositions containing the germ of theories which were capable of independent development. But, speaking generally, the further progress of geometry on general lines was practically barred by the restrictions of method and form which were inseparable from the classical Greek geometry. True, it was open to geometers to discover and investigate curves of a higher order than conics, such as spirals, conchoids, and the like. But the Greeks could not get very far even on these lines in the absence of some system of coordinates and without freer means of manipulation such as are afforded by modern algebra, in contrast to the geometrical algebra, which could only deal with equations connecting lines, areas, and volumes, but involving no higher dimensions than three, except in so far as the use of proportions allowed a very partial exemption from this limitation. The theoretical methods available enabled quadratic, cubic and bi-quadratic equations or their equivalents to be solved. But all the solutions were *geometrical*; in other words, quantities could only be represented by lines, areas and volumes, or ratios between them. There was nothing corresponding to operations with general algebraical quantities irrespective of what they represented. There were no *symbols* for such quantities. In particular, the irrational was discovered in the form of incommensurable *lines*; hence irrationals came to be represented by straight lines as they

quantity, it was only an abbreviation for the word with the meaning of 'an undetermined multitude or not a general quantity. The restriction then of the employed by geometers to the geometrical form of operated as an insuperable obstacle to any really new nature in theoretical geometry.

It might be thought that there was room for further extensions in the region of solid geometry. But the fundamental principles of solid geometry had also been laid down in Books XI-XIII; the theoretical geometry of the sphere had been fully treated in the ancient *sphaeric*; and any application of solid geometry, or of loci in three dimensions, was hampered by the same restrictions of method which hindered the further progress of plane geometry.

Theoretical geometry being thus practically at the end of its resources, it was natural that mathematicians, seeking an opening, should turn to the *applications* of geometry. The obvious branch remaining to be worked out was the geometry of measurement, or *mésuration* in its widest sense, which of course had to wait on pure theory and to be based on experimental results. One species of mensuration was immediately required for astronomy, namely the measurement of triangles, especially spherical triangles; in other words, trigonometry plane and spherical. Another species of mensuration was that in which an example had already been set by Archimedes, namely the measurement of areas and volumes of different shapes by arithmetical approximations to their true values in cases where they involved surds or the ratio (π) between the circumference of a circle and its diameter; the object of this mensuration was largely practical. Of these two kinds of mensuration, the first (trigonometry) is represented by Hipparchus, Menelaus and Ptolemy; the second by Heron of Alexandria. These mathematicians will be dealt with in the next chapters; this chapter will be devoted to the successors of the great geometers who worked on the same lines as the last. Unfortunately we have only very meagre information

s, and especially by Pappus and Eutocius. Some of are very interesting, and it is evident from the facts from the works of such writers as Diocles and Eysodorus that, for some time after Archimedes and Apollonius, mathematicians had a thorough grasp of the contents of the works of the great geometers, and were able to use the principles and methods laid down therein with ease and skill.

Two geometers properly belonging to this chapter have not yet been dealt with. The first is NICOMEDES, the inventor of the conchoid, who was about intermediate in date between Eratosthenes and Apollonius. The conchoid has already been described above (vol. i, pp. 238-40). It gave a general method of solving any $\nu\epsilon\upsilon\sigma\iota\varsigma$ where one of the lines which cut off an intercept of given length on the line verging to a given point is a straight line; and it was used both for the finding of two mean proportionals and for the trisection of any angle, these problems being alike reducible to a $\nu\epsilon\upsilon\sigma\iota\varsigma$ of this kind. How Nicomedes discussed the properties of the curve in itself is not certain; we only know from Pappus that he proved two properties, (1) that the so-called 'ruler' in the instrument for constructing the curve is an asymptote, (2) that any straight line drawn in the space between the 'ruler' or asymptote and the conchoid must, if produced, be cut by the conchoid.¹ The definition of the curve referred to polar coordinates is, as we have seen, $r = a + b \sec \theta$. According to Eutocius, Nicomedes devoted himself inordinately on his discovery of this curve, comparing it with Eratosthenes's mechanism for finding any number of mean proportionals, to which he objected formally that it was of no length on the ground that it was impracticable and wholly outside the spirit of geometry.²

Nicomedes is associated by Pappus with Dinostratus, the teacher of Menaechmus, and others as having applied to the squaring of the circle the curve invented by Hippias and known as the *quadratrix*,³ which was originally intended for the purpose of trisecting any angle. These facts are all that

known as the *cissoïd*, which he used to solve the problem of the two mean proportionals, and also (pp. 47-48) as the author of a method of solving the equation of a certain cubic equation by means of the intersection of an ellipse and a hyperbola. We are indebted to Eutocius¹ for information on both these subjects to Eutocius,¹ and to us that the fragments which he quotes came from a work *περὶ πυρρίων*, *On burning-mirrors*. The connection of the two things with the subject of this treatise is not clear, and we may perhaps infer that it was a work of considerable scope. What exactly were the forms of the burning-mirrors discussed in the treatise it is not possible to say, but it is probably safe to assume that among them were (1) mirrors in the forms (1) of a sphere, (2) of a parabola, (3) of the surface described by the revolution of a circle about its major axis. The author of the *Fragmentum Mathematicum Bobiense* says that Apollonius in his book *On burning-mirror* discussed the case of the concave mirror, showing about what point ignition would take place, and it is certain that Apollonius was aware that an ellipse has the property of reflecting all rays through one focus to the other focus. Nor is it likely that the corresponding property of a parabola with reference to rays parallel to the axis was unknown to Apollonius. Diocles therefore, writing one or more later than Apollonius, could hardly have dealt with all three cases. True, Anthemius (d. A.D. 534) in his fragment on burning-mirrors says that the ancients, while mentioning the usual burning-mirrors, saying that such figures are conic sections, omitted to give the metrical proofs of their properties, and how produced, and gave metrical proofs of their properties. But if the properties were commonly known and quoted, it is obvious that they must have been proved by the ancients, and the explanation of Anthemius's remark is presumably that the original proofs in which they were proved (e.g. those of Apollonius and Diocles) were already lost when he wrote. There can be no trace of Diocles's work left either in Greek

¹ Eutocius, *loc. cit.*, p. 66. 8 sq., p. 160. 3 sq.

we have a fragment from it in the *Fragmentum mathematicum Bobiense*. But Moslem writers regarded Diocles as discoverer of the parabolic burning-mirror; 'the ancients', al Singārī (Sachāwī, Anṣārī), 'made mirrors of plane surfaces. Some made them concave (i.e. spherical) until al-Diūklis showed and proved that, if the surface of mirrors has its curvature in the form of a parabola, they have the greatest power and burn most strongly. There is a work on this subject composed by Ibn al-Haitham.' This work survives in Arabic and in Latin translations, and is edited by Heiberg and Wiedemann¹; it does not, however, mention the name of Diocles, but only those of Archimedes and Anthemius. Ibn al-Haitham says that famous men like Archimedes and Anthemius had used mirrors made of a number of spherical rings; afterwards, he adds, they considered the form of curves which would reflect rays to one point and found that the concave surface of a paraboloid of revolution has this property. It is curious to find Ibn al-Haitham saying that the ancients had not set out the proofs explicitly, nor the method by which they discovered them, which almost exactly recall those of Anthemius himself. Nevertheless the whole course of Ibn al-Haitham's proofs is on the Greek model, Apollonius being actually quoted by name in the proof of the main property of the parabola required, namely that the tangent at any point of the curve makes equal angles with the focal distance of the point and the straight line drawn through it parallel to the axis. A proof of the property actually survives in the Greek *Fragmentum mathematicum Bobiense*, which evidently came from some source on the parabolic burning-mirror; but Ibn al-Haitham does not seem to have had even this fragment at his disposal, for his proof takes a different course, distinguishing three different cases, reducing the property by analysis to the main property $AN = AT$, and then working out the synthesis. The proof in the *Fragmentum* is worth giving. It is

to the axis meeting PT in Y , and join SY .

Now

$$\begin{aligned} PN^2 &= AL \cdot AN \\ &= 4AS \cdot AN \\ &= 4AS \cdot AT \text{ (since } AN = AT \text{)} \end{aligned}$$

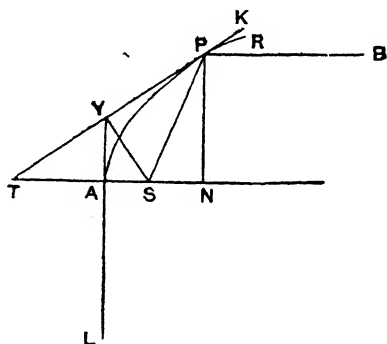
But $PN = 2AY$ (since $AN = AT$);

therefore

$$AY^2 = TA \cdot AS,$$

and the angle TYS is right.

The triangles SYT , SYP being right-angled, and equal to YP , it follows that $SP = ST$.



With the same figure, let BP be a ray parallel to AS impinging on the curve at P . It is required to show that the angles of incidence and reflection (to S) are equal.

We have $SP = ST$, so that 'the angles at the point S are equal. So', says the author, 'are the angles TPA and SPB [the angles between the tangent and the curve on either side of the point of contact]. Let the difference between these angles be taken; therefore the angles SPA , RPB which are [again 'mixed' angles] are equal. Similarly we can show that all the lines drawn parallel to AS will be inclined at equal angles to the point S .'

The author then proceeds: 'Thus burning-mirrors constructed with the surface of impact (in the form of a section of a right-angled cone may easily, in the

is shown, be proved to bring about ignition at the point created.'

Heiberg held that the style of this fragment is Byzantine and that it is probably by Anthemius. Cantor conjectured that here we might, after all, have an extract from Diocles's work. Heiberg's supposition seems to me untenable because of the author's use (1) of the ancient terms 'section of a right-angled cone' for parabola and 'diameter' for axis (say nothing of the use of the parameter, of which there is no word in the genuine fragment of Anthemius), and (2) of the mixed 'angles of contact'. Nor does it seem likely that

Diocles, living a century after Apollonius, would have spoken of the 'section of a right-angled cone' instead of a parabola, or used the 'mixed' angle of which there is only the earliest survival in Euclid. The assumption of the equality of the two angles made by the curve with the tangent on either sides of the point of contact reminds us of Aristotle's assumption of the equality of the angles 'of a segment' of a circle as prior to the truth proved in Eucl. I. 5. I am inclined, therefore, to date the fragment much earlier even than Diocles. Zeuthen suggested that the property of the paraboloidal mirror may have been discovered by Archimedes, and, according to a Greek tradition, wrote *Catoptrica*. This, however, does not receive any confirmation in Ibn al-Haitham or Anthemius, and we can only say that the fragment at least goes back to an original which was probably not later than Apollonius.

PERSEUS is only known, from allusions to him in Proclus,¹ as the discoverer and investigator of the *spiral sections*. They were classified by Proclus among curves obtained by cutting a cylinder, and in this respect they are associated with the conic sections. We may safely infer that they were discovered only after the conic sections, and only after the theory of conics had been considerably developed. This was already the case in Euclid's time, and it is probable, therefore, that Perseus was

like renown as discoverers of other curves to be obtained by cutting well-known solid figures other than the cone or cylinder. A particular case of one such solid figure, the *σπειρα*, had already been employed by Archytas, and the general form of it would not unnaturally be thought likely to give sections worthy of investigation. Since Geometria is Proclus's authority, Perseus may have lived at any date from Euclid's time to (say) 75 B.C., but the most probable suggestion seems to be that he came before Apollonius and not after Euclid in date.

The *spire* in one of its forms is what we call a *tore*, or *anchor-ring*. It is generated by the revolution of a circle about a straight line in its plane in such a way that the circle always passes through the axis of revolution. It takes three forms according as the axis of revolution is (a) altogether outside the circle, when the spire is *dissected* (*διεχής*), (b) a tangent to the circle, when the surface is *sinuous* (*συνεχής*), or (c) a chord of the circle, when it is *interlaced* (*ἐμπεπλεγμένη*), or *crossing-itself* (*ἐπαλλάττουσα*). The alternative name for the surface was *κρίκος*, a *ring*. Perseus celebrated his discovery in an epigram to the effect 'Perseus on his discovery of three lines (curves) upon a circle gave thanks to the gods therefor'.¹ There is no doubt about the meaning of 'three lines upon five sections' (*τρεις γραμμὰς ἐπὶ πέντε τομαῖς*). We gather from Proclus' account of three sections distinguished by Perseus that the plane of section was always parallel to the axis of revolution or perpendicular to the plane which cuts the tore symmetrically like the division in a split-ring. It is difficult to interpret the phrase if it means three curves made by five different sections. Proclus indeed implies that the three curves were sections of the three kinds of tore respectively (the open, the closed, and the interlaced), but this is evidently a mistake. Tannery interprets the phrase as meaning 'three curves in addition to five sections'.² Of these the five sections belong to the open tore, in which the distance between the

dicular distance of the plane of section from the axis of rotation, we can distinguish the following cases :

- (1) $c + a > d > c$. Here the curve is an oval.
- (2) $d = c$: transition from case (1) to the next case.
- (3) $c > d > c - a$. The curve is now a closed curve narrowest in the middle.
- (4) $d = c - a$. In this case the curve is the *hippopede* (horse-fetter), a curve in the shape of the figure of 8. The lemniscate of Bernoulli is a particular case of this curve, that namely in which $c = 2a$.
- (5) $c - a > d > 0$. In this case the section consists of two ovals symmetrical with one another.

The three curves specified by Proclus are those corresponding to (1), (3) and (4).

When the tore is 'continuous' or closed, $c = a$, and we have sections corresponding to (1), (2) and (3) only; (4) reduces to two circles touching one another.

But Tannery finds in the third, the interlaced, form of tore three new sections corresponding to (1) (2) (3), each with an oval in the middle. This would make three curves in addition to the five sections, or eight curves in all. We cannot be certain that this is the true explanation of the phrase in the epigram; but it seems to be the best suggestion that has been made.

According to Proclus, Perseus worked out the property of his curves, as Nicomedes did that of the conchoid, Hippias that of the *quadratrix*, and Apollonius those of the three conic sections. That is, Perseus must have given, in some form, the equivalent of the Cartesian equation by which we can represent the different curves in question. If we refer the tore to three axes of coordinates at right angles to one another with the centre of the tore as origin the axis of z being taken

where c, a have the same meaning as above. The different sections parallel to the axis of revolution are obtained by giving (say) z any value between 0 and $c + a$. For the value $z = a$ the curve is the oval of Cassini which has the property that, if r, r' be the distances of any point on the curve from two fixed points as poles, $rr' = \text{const.}$ For, if $z = a$, the equation becomes

$$(x^2 + y^2 + c^2)^2 = 4c^2x^2 + 4c^2a^2,$$

or
$$\{\overline{c - x^2 + y^2}\} \{\overline{c + x^2 + y^2}\} = 4c^2a^2;$$

and this is equivalent to $rr' = \pm 2ca$ if x, y are the coordinates of any point on the curve referred to Ox, Oy as axes, where O is the middle point of the line ($2c$ in length) joining the two poles, and Ox lies along that line in either direction, while Oy is perpendicular to it. Whether Perseus discussed this case and arrived at the property in relation to the two poles is of course quite uncertain.

Isoperimetric figures.

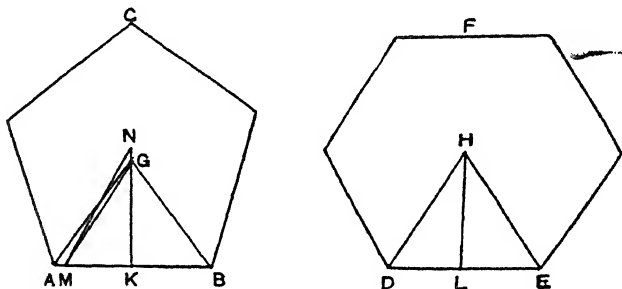
The subject of isoperimetric figures, that is to say, the comparison of the areas of figures having different shapes but the same perimeter, was one which would naturally appeal to the early Greek mathematicians. We gather from Proclus's notes on Eucl. I. 36, 37 that those theorems, proving that parallelograms or triangles on the same or equal bases and between the same parallels are equal in area, appeared to the ordinary person paradoxical because they meant that, by moving the side opposite to the base in the parallelogram, or the vertex of the triangle, to the right or left as far as we please, we may increase the perimeter of the figure to any extent while keeping the same area. Thus the perimeter in parallelograms or triangles is in itself no criterion as to their area. Misconception on this subject was rife among non-mathematicians. Proclus tells us of descriptions of figures which

themselves, so that, while they got a reputation for greater honesty, they in fact took more than their share of the produce.¹ Several remarks by ancient authors show the prevalence of the same misconception. Thucydides estimates the size of Sicily according to the time required for circumnavigating it.² About 130 B.C. Polybius observed that there were people who could not understand that camps of the same periphery might have different capacities.³ Quintilian has a similar remark, and Cantor thinks he may have had in his mind the calculations of Pliny, who compares the size of different parts of the earth by adding their lengths to their breadths.⁴

ZENODORUS wrote, at some date between (say) 200 B.C. and A.D. 90, a treatise *περὶ ἰσομέτρων σχημάτων*, *On isometric figures*. A number of propositions from it are preserved in the commentary of Theon of Alexandria on Book I of Ptolemy's *Syntaxis*; and they are reproduced in Latin in the third volume of Hultsch's edition of Pappus, for the purpose of comparison with Pappus's own exposition of the same propositions at the beginning of his Book V, where he appears to have followed Zenodorus pretty closely while making some changes in detail.⁵ From the closeness with which the style of Zenodorus follows that of Euclid and Archimedes we may judge that his date was not much later than that of Archimedes, whom he mentions as the author of the proposition (*Measurement of a Circle*, Prop. 1) that the area of a circle is half that of the rectangle contained by the perimeter of the circle and its radius. The important propositions proved by Zenodorus and Pappus include the following: (1) *Of all regular polygons of equal perimeter, that is the greatest in area which has the most angles.* (2) *A circle is greater than any regular polygon of equal contour.* (3) *Of all polygons of the same number of sides and equal perimeter the equilateral and equiangular polygon is the greatest in area.* Pappus

area. Zenodorus's treatise was not confined to propositions about plane figures, but gave also the theorem that *Of solid figures the surfaces of which are equal, the sphere is greatest in solid content.*

We will briefly indicate Zenodorus's method of proof. begin with (1); let ABC, DEF be equilateral and equiangular polygons of the same perimeter, DEF having more angles than ABC . Let G, H be the centres of the circumscribed circles, GK, HL the perpendiculars from G, H to the sides AB, DE , so that K, L bisect those sides.



Since the perimeters are equal, $AB > DE$, and $AK > DL$. Make KM equal to DL and join GM .

Since AB is the same fraction of the perimeter that the angle AGB is of four right angles, and DE is the same fraction of the same perimeter that the angle DHE is of four right angles, it follows that

$$AB:DE = \angle AGB:\angle DHE,$$

that is, $AK:MK = \angle AGK:\angle DHL.$

But $AK:MK > \angle AGK:\angle MGK$

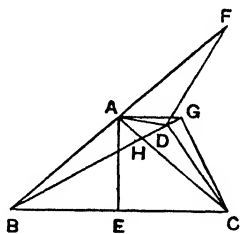
(this is easily proved in a lemma following by the usual method of drawing an arc of a circle with G as centre and GM as radius cutting GA and GK produced. The proposition is course equivalent to $\tan \alpha / \tan \beta > \alpha / \beta$, where $\frac{1}{2}\pi > \alpha > \beta$.)

NA is equal to HL , so that $GA < HL$.

But the area of the polygon ABC is half the rectangle contained by GK and the perimeter, while the area of the polygon DEF is half the rectangle contained by HL and the same perimeter. Therefore the area of the polygon DEF is the greater.

(2) The proof that a circle is greater than any regular polygon with the same perimeter is deduced immediately from Archimedes's proposition that the area of a circle is equal to the right-angled triangle with perpendicular side equal to the radius and base equal to the perimeter of the circle; Zenodorus inserts a proof *in extenso* of Archimedes's proposition, with preliminary lemma. The perpendicular from the centre of the circle circumscribing the polygon is easily proved to be less than the radius of the given circle with perimeter equal to that of the polygon; whence the proposition follows.

(3) The proof of this proposition depends on some preliminary lemmas. The first proves that, if there be two triangles on the same base and with the same perimeter, one being isosceles and the other scalene, the isosceles triangle has the greater area. (Given the scalene triangle BDC on the base BC , it is easy to draw on BC as base the isosceles triangle having the same perimeter. We have only to take BH equal to $\frac{1}{2}(BD + DC)$, bisect BC at E , and erect at E the perpendicular AE such that $AE^2 = BH^2 - BE^2$.)



Produce BA to F so that $BA = AF$, and join AD , DF .

Then $BD + DF > BF$, i.e. $BA + AC$, i.e. $BD + DC$, by hypothesis; therefore $DF > DC$, whence in the triangles FAD , CAD the angle $FAD >$ the angle CAD .

Therefore
$$\angle FAD > \frac{1}{2} \angle FAC$$
$$> \angle BCA.$$

Make the angle FAG equal to the angle BCA or ABC , so that AG is parallel to BC ; let BD produced meet AG in G , and join GC .

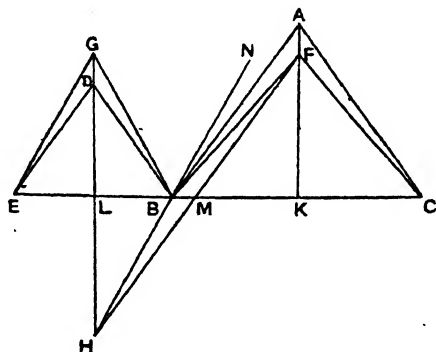
Then

$$\triangle ABC = \triangle GBC$$

$$> \triangle DBC.$$

The second lemma is to the effect that, given two isosceles triangles not similar to one another, if we construct on the same bases two triangles *similar to one another* such that the sum of their perimeters is equal to the sum of the perimeters of the first two triangles, then the sum of the areas of the similar triangles is greater than the sum of the areas of the non-similar triangles. (The easy construction of the similar triangles is given in a separate lemma.)

Let the bases of the isosceles triangles, EB, BC , be placed on one straight line, BC being greater than EB .



Let ABC, DEB be the similar isosceles triangles, and FE, GE, B the non-similar, the triangles being such that

$$BA + AC + ED + DB = BF + FC + EG + GB.$$

Produce AF, GD to meet the bases in K, L . Then clearly AK, GL bisect BC, EB at right angles at K, L .

Produce GL to H , making LH equal to GL .

Join HB and produce it to N ; join HF .

Now, since the triangles ABC, DEB are similar, the angle ABC is equal to the angle DEB or DBE .

ow, by hypothesis, $DB + BA = GB + BF$;

fore $DB + BA = HB + BF > HF$.

an easy lemma, since the triangles DEB , ABC are similar,

$$\begin{aligned}(DB + BA)^2 &= (DL + AK)^2 + (BL + BK)^2 \\ &= (DL + AK)^2 + LK^2.\end{aligned}$$

efore $(DL + AK)^2 + LK^2 > HF^2$

$$> (GL + FK)^2 + LK^2,$$

ce $DL + AK > GL + FK$,

t follows that $AF > GD$.

t $BK > BL$; therefore $AF \cdot BK > GD \cdot BL$.

nce the 'hollow-angled (figure)' (*κοιλογώνιον*) $ABFC$ is
er than the hollow-angled (figure) $GEDB$.

ding $\triangle DEB + \triangle BFC$ to each, we have

$$\triangle DEB + \triangle ABC > \triangle GEB + \triangle FBC.$$

e above is the only case taken by Zenodorus. The proof
holds if $EB = BC$, so that $BK = BL$. But it fails in the
n which $EB > BC$ and the vertex G of the triangle EB
ging to the non-similar pair is still above D and not
y it (as F is below A in the preceding case). This was
oubt the reason why Pappus gave a proof intended to
y to all the cases without distinction. This proof is the
as the above proof by Zenodorus up to the point where
proved that

$$DL + AK > GL + FK,$$

here diverges. Unfortunately the text is bad, and gives
fficient indication of the course of the proof; but it would
that Pappus used the relations

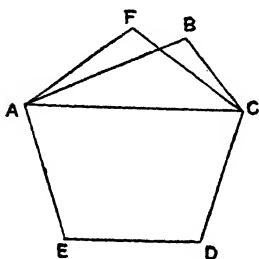
$$DL : GL = \triangle DEB : \triangle GEB,$$

$$AK : FK = \triangle ABC : \triangle FBC,$$

$$AK^2 : DL^2 = \triangle ABC : \triangle DEB,$$

indicates that he will give later, but in the meantime the promise is not fulfilled.

Then follows the proof that the maximum polygon of given perimeter is both equilateral and equiangular.



(1) It is equilateral.

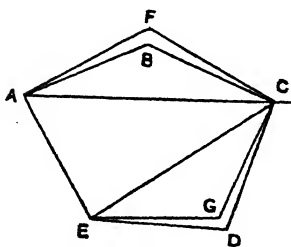
For, if not, let two sides of the maximum polygon, as AB and BC , be unequal. Join AC , and on the base AC draw the isosceles triangle AFB such that $AF + FC = AB$. The area of the triangle AFB

is greater than the area of the triangle ABC , and the whole polygon has been increased by this substitution: which is impossible, as by hypothesis the polygon is of maximum area.

Similarly it can be proved that no other side is greater than any other.

(2) It is also equiangular.

For, if possible, let the maximum polygon $ABCDE$



we have proved to be of maximum area. Suppose we have the angle at B greater than the angle at D . Then construct two non-similar isosceles triangles AFB and CED on the bases AC , CE as bases described on the sides AB , CD of the polygon. The isosceles triangles FAF and CEC are similar to one another which implies that the sum of their perimeters is less than the sum of the perimeters of the original polygon.

DEC . Then the sum of the areas of the two similar triangles is greater than the sum of the areas of the triangles BAC , DEC ; the area of the polygon is therefore increased, which is contrary to the hypothesis.

Hence no two angles of the polygon can be unequal.

The maximum polygon of given perimeter is therefore equilateral and equiangular.

Dealing with the sphere in relation to other s

surfaces equal to that of the sphere, Zenodorus confined himself to proving (1) that the sphere is greater if the other with surface equal to that of the sphere is a solid formed by the revolution of a regular polygon about a diameter joining it as in Archimedes, *On the Sphere and Cylinder*, I, and (2) that the sphere is greater than any of the regular solids having its surface equal to that of the sphere.

Proclus's treatment of the subject is more complete in that he proves that the sphere is greater than the cone or cylinder the surface of which is equal to that of the sphere, and further that of the five regular solids which have the same surface the one which has more faces is the greater.¹

HYPsicLES (second half of second century B.C.) has already been mentioned (vol. i, pp. 419-20) as the author of the conclusion of the *Elements* known as Book XIV. He is quoted by Proclus as having given a definition of a polygonal number as follows:

There are as many numbers as we please beginning from one and increasing by the same common difference, then, when the common difference is 1, the sum of all the numbers is a triangular number; when 2, a square; when 3, a pentagonal number [and so on]. And the number of angles is called the number which exceeds the common difference by 2, the side after the number of terms including 1.'

This definition amounts to saying that the n th a -gonal number (counting as the first) is $\frac{1}{2}n\{2 + (n-1)(a-2)\}$. If, as is probable, Hypsicles wrote a treatise on polygonal numbers, it has not survived. On the other hand, the *Ἀναφορικός* (*Ascent*) known by his name has survived in Greek as well as in Latin, and has been edited with translation.² True, the work (if it really be by Hypsicles, and not a clumsy effort of a beginner working from an original by Hypsicles) does not do credit to its author; but it is in some respects

work in which we find the division of the ecliptic circle into 360 'parts' or degrees. The author says, after the preliminary propositions,

'The circle of the zodiac having been divided into 360 equal circumferences (arcs), let each of the latter be called a *degree in space* (μοῖρα τοπική, 'local' or 'spatial part'). And similarly, supposing that the time in which the zodiac circle returns to any position it has left is divided into 360 equal times, let each of these be called a *degree in time* (μοῖρα χρονική).'

From the word καλείσθω ('let it be called') we may perhaps infer that the terms were new in Greece. This brings up the question of the origin of the division (1) of the circle of the zodiac, (2) of the circle in general, into 360 parts. On this question innumerable suggestions have been made. With reference to (1) it was suggested as long ago as 1788 (by F. B. de Maleoni) that the division was meant to correspond to the number of days in the year. Another suggestion is that it would early be discovered that, in the case of any circle inscribed hexagon dividing the circumference into six parts, each has each of its sides equal to the radius, and that this would naturally lead to the circle being regularly divided into 60 parts; after this, the very ancient sexagesimal system would naturally come into operation and each of the parts would be divided into 60 subdivisions, giving 360 of these for the whole circle. Again, there is an explanation which is not exactly geometrical, namely that in the Babylonian numeral system, which combined the use of 6 and 10 as bases, the numbers 60, 360, 3600 were fundamental round numbers, and these numbers were transferred from arithmetic to the heavens. The obvious objection to the first of these explanations (referring the 360 to the number of days in the solar year) is that the Babylonians were well acquainted, as far back as the monuments go, with 365.2 as the number of days in the year. A variant of the hexagon-theory is the suggestion that

al to divide each of the sixth parts into either 10 or 60; the former division would account for the attested on of the day into 60 hours, while the latter division on xagesimal system would give the 360 time-degrees (each minutes) making up the day of 24 hours. The purely netical explanation is defective in that the series of ers for which the Babylonians had special names is not 0, 3600 but 60 (Soss), 600 (Ner), and 3600 or 60^2 (Sar). he whole, after all that has been said, I know of no suggestion than that of Tannery.¹ It is certain that he division of the ecliptic into 360 degrees and that of $\nu\chi\theta\acute{\eta}\mu\epsilon\rho\omicron\nu$ into 360 time-degrees were adopted by the s from Babylon. Now the earliest division of the c was into 12 parts, the signs, and the question is, how the signs subdivided? Tannery observes that, accord- the cuneiform inscriptions, as well as the testimony of authors, the sign was divided into parts one of which (*atu*) was double of the other (*murran*), the former being h, the other (called *stadium* by Manilius) $1/60$ th, of the the former division would give 360 parts, the latter 720 for the whole circle. The latter division was more al, in view of the long-established system of sexagesimal ons; it also had the advantage of corresponding toler- closely to the apparent diameter of the sun in comparison he circumference of the sun's apparent circle. But, on her hand, the double fraction, the $1/30$ th, was contained circle of the zodiac approximately the same number of as there are days in the year, and consequently corre- ed nearly to the distance described by the sun along the in one day. It would seem that this advantage was ent to turn the scale in favour of dividing each sign of odia into 30 parts, giving 360 parts for the whole

While the Chaldaeans thus divided the ecliptic into arts, it does not appear that they applied the same divi- to the equator or any other circle. They measured angles eral by *ells* an ell representing 2° so that the complete

the *diameter* of the circle into 60 ells in accordance with the usual sexagesimal division, and then came to divide the circumference into 180 such ells on the ground that the circumference is roughly three times the diameter. The measurement in *ells* and *dactyli* (of which there were 24 to the ell) survives in Hipparchus (*On the Phaenomena of Eudoxus and Aratus*), and some measurements in terms of the same are given by Ptolemy. It was Hipparchus who first divided the circle in general into 360 parts or degrees, and the introduction of this division coincides with his invention of trigonometry.

The contents of Hypsicles's tract need not detain us. The problem is: If we know the ratio which the length of the longest day bears to the length of the shortest day at any given place, to find how many time-degrees it takes any sign to rise; and, after this has been found, the author finds what length of time it takes any given degree in any sign to rise, i.e. the interval between the rising of one degree-point of the ecliptic and that of the next following. It is explained that the longest day is the time during which one half of the zodiac (Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius) rises, and the shortest day the time during which the other half (Capricornus, Aquarius, Pisces, Aries, Taurus, Gemini) rises. Now at Alexandria the longest day is to the shortest as 13 to 5; the longest therefore contains 210 'time-degrees' and the shortest 150. The two quadrants Cancer-Virgo and Libra-Sagittarius take the same time to rise, namely 105 'time-degrees', and the two quadrants Capricornus-Pisces and Aries-Gemini each take the same time, namely 75 'time-degrees'. It is further assumed that the times taken by Virgo, Cancer, Gemini, Taurus, Aries are in descending arithmetical progression, while the times taken by Libra, Scorpio, Sagittarius, Capricornus, Aquarius, Pisces continue the same ascending arithmetical series. The following lemmas are proved:

I. If α and β

If $a_1, a_2 \dots a_n \dots a_{2n-1}$ is a descending arithmetical progression of $2n-1$ terms with δ as common difference and a_n the middle term, then

$$a_1 + a_2 + \dots + a_{2n-1} = (2n-1)a_n.$$

If $a_1, a_2 \dots a_n, a_{n+1} \dots a_{2n}$ is a descending arithmetical progression of $2n$ terms, then

$$a_1 + \dots + a_{2n} = n(a_1 + a_{2n}) = n(a_2 + a_{2n-1}) = \dots$$

$$= n(a_n + a_{n+1}).$$

Now let A, B, C be the descending series the sum of which is 105, and D, E, F the next three terms in the same series the sum of which is 75, the common difference being δ ; we have, by (I),

$$A + B + C - (D + E + F) = 9\delta, \text{ or } 30 = 9\delta,$$

$$\delta = 3\frac{1}{3}.$$

Next, by (II), $A + B + C = 3B$, or $3B = 105$, and $B = 35$;

therefore A, B, C, D, E, F are equal to $38\frac{1}{3}, 35, 31\frac{2}{3}, 28\frac{1}{3}, 25, 21\frac{2}{3}$ time-degrees respectively, which the author of the tract expresses in time-degrees and minutes as $38^t 20', 35^t, 31^t 40', 28^t 20', 25^t, 21^t 40'$. We have now to carry through the same procedure for each degree in each sign. If the difference between the times taken to rise by one sign and the next is $20'$, what is the difference for each of the 30 degrees in each sign? We have here 30 terms followed by 30 other terms in the same descending arithmetical progression, and the formula (I) gives $3^t. 20' = (30)^2 d$, where d is the common difference; therefore $d = \frac{1}{900} \times 3^t. 20' = 0^t 0' 13'' 20'''$. Lastly, the sign corresponding to $21^t 40'$. This is the sum of a descending arithmetical progression of 30 terms $a_1, a_2 \dots a_{30}$ with common difference $0^t 0' 13'' 20'''$. Therefore, by (III), $21^t 40' = 15(a_1 + a_{30})$, whence $a_1 + a_{30} = 1^t 26' 40''$. Now,

the times corresponding to all the degrees in the found.

The procedure was probably, as Tannery thought direct from the Babylonians, who would no doubt have the purpose of enabling the time to be determined at any hour of the night. Another view is that the method was astrological rather than astronomical (Manitius). In any case the method was exceedingly rough, and the errors of increases and decreases in the times of the risings of the signs in arithmetical progression are not in accordance with the facts. The book could only have been written by the invention of trigonometry by Hipparchus, for the method of finding the times of rising of the signs is really spherical trigonometry, and these times were calculated by Hipparchus and Ptolemy by means of chords.

DIONYSODORUS is known in the first place as the author of a solution of the cubic equation subsidiary to the problem of Archimedes, *On the Sphere and Cylinder*, II. 4, To cut a sphere by a plane so that the volumes of the segments cut off from one another a given ratio (see above, p. 46). Upon this Dionysodorus was supposed to be Dionysodorus of Pontus, whom Suidas describes as 'a mathematician of mention in the field of education'. But we now have a fragment of the Herculanum Roll, No. 1044, that was a pupil, first of Eudemus, and afterwards of Dionysodorus the son of Dionysodorus the Caunian'. Now it is evident that Eudemus of Pergamum to whom Apollonius dedicated the first two Books of his *Conics*, and Apollonius asks him to show Book II to Philonides. In another fragment Philonides is said to have published some work of Dionysodorus. Hence our Dionysodorus may be Dionysodorus of Caunus and a contemporary of Apollonius, or a little later.¹ A Dionysodorus is also mentioned by Heron as the author of a tract *On the Spire* (or tore) in which he shows that, if d be the diameter of the revolving circle,

¹ W. Schmidt in *Bibliotheca mathematica*, iv, pp. 3.

² Heron, *Metrica*, ii. 13, p. 128. 3.

es the tore, and c the distance of its centre from the revolution,

$$(\text{volume of tore}) : \pi c^2 \cdot d = \frac{1}{4} \pi d^2 : \frac{1}{2} cd,$$

$$(\text{volume of tore}) = \frac{1}{2} \pi^2 \cdot cd^2,$$

is of course the product of the area of the generating and the length of the path of its centre of gravity. The which the result is stated, namely that the tore is to under with height d and radius c as the generating of the tore is to half the parallelogram cd , indicates clearly that Dionysodorus proved his result by the same as that employed by Archimedes in the *Method* and book *On Conoids and Spheroids*; and indeed the proof imimedean lines is not difficult.

re passing to the mathematicians who are identified the discovery and development of trigonometry, it will venient, I think, to dispose of two more mathematicians ng to the last century B.C., although this involves t departure from chronological order; I mean Posidonius minus.

DONIUS, a Stoic, the teacher of Cicero, is known as ius of Apamea (where he was born) or of Rhodes he taught); his date may be taken as approximately . B.C. In pure mathematics he is mainly quoted as the of certain definitions, or for views on technical terms, eorem' and 'problem', and subjects belonging to ele- y geometry. More important were his contributions hematical geography and astronomy. He gave his work on geography the title *On the Ocean*, using the hich had always had such a fascination for the Greeks; tents are known to us through the copious quotations t in Strabo; it dealt with physical as well as mathe- l geography, the zones, the tides and their connexion e moon, ethnography and all sorts of observations made

Posidonius also wrote a separate tract on the size of the earth. The two things which are sufficiently important to mention here are (1) Posidonius's measurement of the circumference of the earth, (2) his hypothesis as to the distance of the sun.

(1) He estimated the circumference of the earth in the same way. He assumed (according to Cleomedes¹) that, the star Canopus, invisible in Greece, was just seen to go below the horizon at Rhodes, rising and setting again immediately at Alexandria. The meridian altitude of the same star at Alexandria was 'one part of a sign, that is, one forty-eighth part of the circumference of the circle' ($= 7\frac{1}{2}^\circ$); and he observed that the distance between the two places (supposed to lie on the same meridian) was 'considered to be 5,000 stades'. The circumference of the earth was thus made out to be 240,000 stades. Unfortunately the estimate of the difference of latitude, $7\frac{1}{2}^\circ$, was very far from correct, the true difference being $5\frac{1}{4}^\circ$ only; moreover the estimate of 5,000 stades for the distance was incorrect, being only the maximum estimate put upon it by many geographers, while some put it at 4,000 and Eratosthenes, by observing the shadows of gnomons, found it to be 3,750 stades. Strabo, on the other hand, says that Posidonius favoured the latest of the measurements which gave the smallest dimensions to the earth, namely about 180,000 stades.² This is evidently 48 times 3,750, so that Posidonius combined Eratosthenes's figure of 3,750 stades with the incorrect estimate of $7\frac{1}{2}^\circ$ for the difference of latitude, although Eratosthenes presumably obtained the figure of 3,750 stades from his estimate (250,000 or 252,000) of the circumference of the earth, combined with an estimate of the difference of latitude which was about $5\frac{2}{5}^\circ$ and therefore near the truth.

(2) Cleomedes³ tells us that Posidonius supposed the sun to move in a circle in which the sun apparently moves round the earth, the radius of which was 10,000 times the size of a circular section of the earth taken through its centre, and that with this assumption he combined

¹ Cleomedes, *De motu circulari*, i. 10, pp. 92-4.

² Strabo, ii. c. 95.

³ Cleomedes, *op. cit.* ii. 1, pp. 144-6, p. 98. 1-5.

ment of Eratosthenes (based apparently upon hearsay) at Syene, which is under the summer tropic, and throughout a circle round it of 300 stades in diameter, the right gnomon throws no shadow at noon. It follows from that the diameter of the sun occupies a portion of the circle 3,000,000 stades in length; in other words, the diameter of the sun is 3,000,000 stades. The assumption that the sun's circle is 10,000 times as large as a great circle of the earth was presumably taken from Archimedes, who had proved in the *Sand-reckoner* that the diameter of the sun's orbit is more than 10,000 times that of the earth; Posidonius in fact took this as the maximum value to be the true value; but his estimate of the sun's size is far nearer the truth than the estimates of Aristarchus, Hipparchus, and Ptolemy. Expressed in terms of the mean diameter of the earth, the estimates of these astronomers give for the diameter of the sun the figures $6\frac{3}{4}$, and $5\frac{1}{2}$ respectively; Posidonius's estimate gives $39\frac{1}{4}$, the figure being 108.9.

In elementary geometry Posidonius is credited by Proclus with certain definitions. He defined 'figure' as 'confining' (*πέρας συγκλειόν*)¹ and 'parallels' as 'those lines which, in one plane, neither converge nor diverge, but have all perpendiculars equal which are drawn from the points of one line to the other'.² (Both these definitions are included in the *Definitions* of Heron.) He also distinguished seven classes of quadrilaterals, and had views on the distinction between *theorem* and *problem*. Another indication of his interest in the fundamentals of elementary geometry is the fact that he wrote a separate work in refutation of the Cyprian Zeno of Sidon, who had objected to the very beginning of the *Elements* on the ground that they contained unproved assumptions. Thus, said Zeno, even Eucl. I. 1 requires it to be admitted that 'two straight lines cannot have a common endpoint'; and, as regards the 'proof' of this fact deduced from the bisection of a circle by its diameter, he would object that it has to be assumed that two arcs of circles cannot have

principles, and he intended by means of these to destroy the whole of geometry.¹ We can understand, therefore, that the tract of Posidonius was a serious work.

A definition of the centre of gravity by one 'Stoic' is quoted in Heron's *Mechanics*, but, as the author goes on to say that Archimedes introduced a further definition, we may fairly assume that the Posidonius in question was the Posidonius of Rhodes, but another, perhaps Philon of Alexandria, a pupil of Zeno of Cittium in the second century B.C.

We now come to GEMINUS, a very important astronomer. Many questions belonging to the history of mathematics are shown by the numerous quotations from him in the *Commentary on Euclid, Book I*. His date and birthplace are uncertain, and the discussions on the subject now in the literature for which reference must be made to the 1898 edition of the so-called *Gemini elementa astronomica* (Leipzig, 1898) and the article 'Geminus' in Pauly-Wissowa's *Encyclopädie*. The doubts begin with his name. Some have included the treatise mentioned in his *Uranologia* (1630), took it to be the Latin Geminus. Manitius, the editor, satisfied himself that it was Geminus, a name judging from the fact that it consistently appears with the properispomenon accent in Greek (*Γεμῖνος*), which is found in inscriptions with the spelling *Γεμείνιος*, which suggests the derivation from *γεμ*, as *Ἐργῖνος* from *ἔργ*, *Ἀλεξῖνος* from *ἀλέξ*; he compares also the Greek names *Ἰκτῖνος*, *Κρατῖνος*. Now, however, Manitius (by Tittel) that the name is, after all, the Latin *Geminus*, that *Γεμῖνος* came to be so written through confusion with *Ἀλεξῖνος*, &c., and that *Γε[μ]εῖνος*, if that is correct, is also wrongly formed on the model of *Ἀγριππείνα*. The occurrence of a Latin name in a work of Greek culture need not surprise us, since Rome had such centres in large numbers during the last century B.C. Geminus, however, in spite of his name, was thoroughly

¹ Proclus on Eucl. I, pp. 214. 18-215. 13, p. 216. 10-19,

upper limit for his date is furnished by the fact that he wrote a commentary on or exposition of Posidonius's work *ὑπερμετέωρων*; on the other hand, Alexander Aphrodisiensis (d. A.D. 210) quotes an important passage from an 'epitome' of Posidonius' *ἐξήγησις* by Geminus. The view most generally held is that he was a Stoic philosopher, born probably on the island of Rhodes, and a pupil of Posidonius, and that he died about 73-67 B.C.

Geminus's works that which has most interest for us is his comprehensive work on mathematics. Proclus, though he makes great use of it, does not mention its title, unless, indeed, in the passage where, after quoting from Geminus on the classification of lines which never meet, he says 'these works I have selected from the *φιλοκαλία* of Geminus',¹ the word *φιλοκαλία* is a title or an alternative title. Pappus, however, quotes a work of Geminus 'on the classification of mathematics' (*ἐν τῷ περὶ τῆς τῶν μαθημάτων τάξεως*), and Eutocius quotes from 'the sixth book of the doctrine of mathematics' (*ἐν τῷ ἕκτῳ τῆς τῶν μαθημάτων θεωρίας*). The former title corresponds well enough to the long extract on the division of the mathematical sciences into arithmetic, geometry, mechanics, astronomy, optics, geodesy, canonic (musical harmony) and logistic which Proclus gives in his prologue, and also to the fragments contained in the *synymi variae collectiones* published by Hultsch in his edition of Heron; but it does not suit most of the other passages borrowed by Proclus. The correct title was most probably that given by Eutocius, *The Doctrine*, or *Theory*, *of the Mathematics*; and Pappus probably refers to one particular section of the work, say the first Book. If the first Book treated of conics, as we may conclude from Eutocius's reference, there must have been more Books to follow; for Proclus has preserved us details about higher mathematics, which must have come later. If again Geminus treated his work and wrote with the same fullness about the various branches of mathematics as he did about geometry,

quotations of Proclus from Geminus's work as in
 alone; we have other collections of extracts, some more
 some less extensive, and showing varieties of tradition
 ing to the channel through which they came down
 scholia to Euclid's *Elements*, Book I, contain a considerable
 part of the commentary on the Definitions of Book I, and
 valuable in that they give Geminus pure and simple, while
 Proclus includes extracts from other authors. Extracts from
 Geminus of considerable length are included in the
 commentary by an-Nairizī (about A.D. 900) who goes
 through the medium of Greek commentaries on
 especially that of Simplicius. It does not appear
 doubted any longer that 'Aganis' in an-Nairizī is
 Geminus; this is inferred from the close agreement between
 an-Nairizī's quotations from 'Aganis' and the corresponding
 passages in Proclus; the difficulty caused by the fact
 that Simplicius calls Aganis 'socius noster' is met by the
 suggestion that the particular word *socius* is either the
 result of the double translation from the Greek or means
 nothing more, in the mouth of Simplicius, than 'collaborer'
 in the sense of a worker in the same field, or 'author'.
 A few extracts again are included in the *Anonymi
 collectiones* in Hultsch's *Heron*. Nos. 5-14 give definitions of
 geometry, logistic, geodesy and their subject-matter, remarks
 on bodies as continuous magnitudes, the three dimensions,
 'principles' of geometry, the purpose of geometry, and remarks
 on optics, with its subdivisions, optics proper, *Catoptrics*,
σκηνογραφική, scene-painting (a sort of perspective), with the
 fundamental principles of optics, e.g. that all light travels
 along straight lines (which are broken in the cases of reflection
 and refraction), and the division between optics and natural
 philosophy (the theory of light), it being the province of the
 latter to investigate (what is a matter of indifference to optics)
 whether (1) visual rays issue from the eye, (2) images pass
 from the object and impinge on the eye, or (3) the intervening
 air is aligned or compacted with the beam-like breeze of
 emanation from the eye.

Nos. 80-6 again in the same collection give the Peripatetic
 explanation of the name mathematics, adding that the

applied by the early Pythagoreans more particularly geometry and arithmetic, sciences which deal with the pure, eternal and the unchangeable, but was extended by later writers to cover what we call 'mixed' or applied mathematics, which, though theoretical, has to do with sensible objects, e.g. astronomy and optics. Other extracts from Geminus are found in certain manuscripts in connexion with Damianus's treatise on conics (published by R. Schöne, Berlin, 1897). The definitions of logistic and geometry also appear, but with decided differences, in the scholia to Plato's *Charmides* 165 E. Lastly, several extracts appear in Eutocius, (1) a remark reproduced in the commentary on Archimedes's *Plane Equilibriums* to the effect that Archimedes in that work gave the name of axioms to what are really axioms, (2) the statement that at Apollonius's time the conics were produced by cutting right cones (right-angled, acute-angled, and obtuse-angled) with planes perpendicular in each case to a generator.¹

The object of Geminus's work was evidently the examination of the first principles, the logical building up of mathematics on the basis of those admitted principles, and the defence of the whole structure against the criticisms of the enemies of the science, the Epicureans and Sceptics, some of whom questioned the unproved principles, and others the logical validity of the deductions from them. Thus in his history Geminus dealt first with the principles or hypotheses (*ἀρχαί, ὑποθέσεις*) and then with the logical deductions, the theorems and problems (*τὰ μετὰ τὰς ἀρχάς*). The distinction between the things which must be taken for granted but are incapable of proof and the things which must not be taken for granted but are matter for demonstration. The principles consisting of definitions, postulates, and axioms, Geminus subjected them severally to a critical examination from this point of view, distinguishing carefully between postulates and axioms, and discussing the legitimacy or otherwise of those postulated by Euclid in each class. In his notes on the definitions Geminus treated them historically, giving the various

of the different species of the thing defined. Thus in the case of 'lines' (which include curves) he distinguishes, first, the composite (e.g. a broken line forming an angle) and the incomposite. The incomposite are subdivided into those 'forming a figure' (*σχηματοποιεῖσαι*) or determinate (e.g. circle, ellipse, cissoid) and those not forming a figure, into terminate and extending without limit (e.g. straight line, parabola, hyperbola, conchoid). In a second classification incomposite lines are divided into (1) 'simple', namely the circle and straight line, the one 'making a figure', the other extending without limit, and (2) 'mixed'. 'Mixed' lines again are divided into (a) 'lines in planes', one kind being a line meeting itself (e.g. the cissoid) and another a line extending without limit, and (b) 'lines on solids', subdivided into lines formed by *sections* (e.g. conic sections, *spiral* curves) and 'lines round solids' (e.g. a helix round a cylinder, sphere, cone, the first of which is uniform, homoeomeric, alike in all its parts, while the others are non-uniform). Geminus gave a corresponding division of surfaces into simple and mixed, the former being plane surfaces and spheres, while examples of the latter are the tore or anchor-ring (though formed by the revolution of a circle about an axis) and the conicoids by revolution (the right-angled conoid, the obtuse-angled conoid) and the two spheroids, formed by the revolution of a parabola, a hyperbola, and an ellipse respectively about their axes). He observes that, while there are three *homoeomeric* or uniform 'lines' (the straight line, the circle, and the cylindrical helix), there are only two homoeomeric surfaces, the plane and the sphere. Other classifications are those of 'angles' (according to the nature of the two lines or curves which form them) and of figures and plane figures.

When Proclus gives definitions, &c., by Posidonius, it is evident that he obtained them from Geminus's work. Such are Posidonius's definitions of 'figure' and 'parallels', and his division of quadrilaterals into seven kinds. We may assume further that, even where Geminus did not mention the name of Posidonius, he was at all events following his lead.

Attempt to prove the Parallel-Postulate.

Geminus devoted much attention to the distinction between theorems and axioms, giving the views of earlier philosophers and mathematicians (Aristotle, Archimedes, Euclid, Ptolemy, the Stoics) on the subject as well as his own. It is important in view of the attacks of the Epicureans and others on mathematics, for (as Geminus says) it is as futile to attempt to prove the indemonstrable (as Apollonius did when he tried to prove the axioms) as it is incorrect to assume that it really requires proof, 'as Euclid did in the fourth postulate [that all right angles are equal] and in the fifth postulate [the parallel-postulate]'.¹

The fifth postulate was the special stumbling-block. Geminus observed that the converse is actually proved by Euclid in I. 17; also that it is conclusively proved that an angle equal to a right angle is not necessarily itself a right angle, e.g. the 'angle' between the circumferences of two semi-circles on two equal straight lines with a common extremity (the right angles to one another); we cannot therefore admit that the converses are incapable of demonstration.² And

we have learned from the very pioneers of this science not to regard 'to mere plausible imaginings when it is a question of the reasonings to be included in our geometrical science. As Aristotle says, it is as justifiable to ask scientists for proofs from a rhetorician as to accept mere plausibilities from a geometer... So in this case (that of the parallel-postulate) the fact that, when the right angles are lessened, the lines converge is true and necessary; but the statement that, since they converge more and more as they are lessened, they will sometime meet is plausible but not necessary in the absence of some argument showing that this is the case of straight lines. For the fact that some lines which approach indefinitely but yet remain non-secant (ἀπώτοι), although it seems improbable and paradoxical, is nevertheless true and fully ascertained with reference to species of lines [the hyperbola and its asymptote and

straight lines which happens in the case of the lines referred to? Indeed, until the statement in the postulate is clinched by proof, the facts shown in the case of the other lines may direct our imagination the opposite way. And, though the controversial arguments against the meeting of the straight lines should contain much that is surprising, is there not the more reason why we should expel from our body of doctrine this merely plausible and unreasoned (hypothetical) theorem, and that it is alien to the special character of the postulates.'¹

Much of this might have been written by a modern geometer. Geminus's attempted remedy was to substitute a definition of parallels like that of Posidonius, based on the notion of *equidistance*. An-Nairizī gives the definition as follows: 'Parallel straight lines are straight lines situated in the same plane and such that the distance between them when they are produced without limit in both directions at the same time, is everywhere the same', to which Geminus adds the statement that the said distance is the shortest straight line that can be drawn between them. Starting from this Geminus proved to his own satisfaction the propositions of Euclid regarding parallels and finally the parallel-postulate. He first gave the propositions (1) that the 'distance' between the two lines as defined is perpendicular to both, and (2) that if a straight line is perpendicular to each of two straight lines and meets both, the two straight lines are parallel, and 'distance' is the intercept on the perpendicular (proved by *reductio ad absurdum*). Next come (3) Euclid's proposition I. 27, 28 that, if two lines are parallel, the alternate angles made by any transversal are equal, &c. (easily proved by drawing the two equal 'distances' through the points of intersection with the transversal), and (4) Eucl. I. 29, the converse of I. 28, which is proved by *reductio ad absurdum*, by means of (2) and (3). Geminus still needs Eucl. I. 30, (about parallels) and I. 33, 34 (the first two propositions relating to parallelograms) for his foundation of the postulate.

are parallel to LV , and Geminus assumes that they are coincident (in other words, that through a given point only one parallel can be drawn to a given straight line), an assumption known as Playfair's Axiom, though it is not stated in Proclus on Eucl. I. 31).

The proof therefore, apparently ingenious as it may seem, goes down. Indeed the method is unsound from the start, since (as Saccheri pointed out), before even the device of drawing parallels by Geminus can be used, it has to be proved that 'the geometrical locus of points equidistant from a given line is a straight line', and this cannot be proved without the use of a postulate. But the attempt is interesting as the first of its kind has come down to us, although there must have been many others by geometers earlier than Geminus.

Coming now to the things which follow from the *Porisms* ($\tauὰ μετὰ τὰς ἀρχάς$), we gather from Proclus that Geminus carefully discussed such generalities as the nature of the *Porisms*, the different views which had been held of the relation between theorems and problems, the nature and scope of *διορισμοί* (conditions and limits of possibility), the nature of 'porism' in the sense in which Euclid used the word, the *Porisms* as distinct from its other meaning of 'corollaries', the different sorts of problems and theorems, the two views of the *Porisms* (complete and partial), *topical* or *locus* problems with the classification of loci. He discussed also philosophical questions, e.g. the question whether a line is made up of indivisible parts ($ἐξ ἀμερῶν$), which came up in connection with Eucl. I. 10 (the bisection of a straight line).

The book was evidently not less exhaustive in its treatment of higher geometry. Not only did Geminus mention the various curves, conchoids and cissoids in his classification, but he showed how they were obtained, and gave proofs of their principal properties. Similarly he gave a proof that there are three homoeomeric or uniform curves, the straight line, the circle and the cylinder. The proof of 'uniformity' (the property that any part of the line or curve will coincide with any other part of the same length) was preceded by a proof that, if two lines be drawn from any point to meet a uniform line

make equal angles with it, the straight lines are equal.¹ Apollonius wrote on the cylindrical helix and proved the uniformity, we may fairly assume that Geminus here drawing upon Apollonius.

ough has been said to show how invaluable a source of information Geminus's work furnished to Proclus and all others on the history of mathematics who had access to it.

In astronomy we know that Geminus wrote an *ἐξήγησις* of Apollonius's work, the *Meteorologica* or *περὶ μετεώρων*. This is the source of the famous extract made from Geminus by Theophrastus of Eresus, and reproduced by Simplicius in his commentary on the *Physics* of Aristotle,² on which Schiavi relied in his attempt to show that it was Heraclides of Pontus, not Aristarchus of Samos, who first put forward the heliocentric hypothesis. The extract is on the distinction between physical and astronomical inquiry as applied to the heavens. It is the business of the physicist to consider the distance of the heaven and stars, their force and quality, their coming into being and decay, and he is in a position to know the facts about their size, shape, and arrangement; astronomy, on the other hand, ignores the physical side, regarding the arrangement of the heavenly bodies by considerations based on the view that the heaven is a real *κόσμος*, and, as such, it tells us of the shapes, sizes and distances of the earth, sun and moon, of eclipses and conjunctions, and of the quality and extent of the movements of the heavenly bodies, it is concerned with the mathematical investigation of quantity, form, or shape, and uses arithmetic and geometry to draw its conclusions. Astronomy deals, not with causes, but with facts; hence it often proceeds by hypotheses, stating expedients by which the phenomena may be saved. For example, why do the sun, the moon and the planets appear to move irregularly? To explain the observed facts we may assume, for instance, that the orbits are eccentric or that the stars describe epicycles on a carrying sphere, and then we have to go farther and examine other

of Pontus] coming forward and saying that, even on the assumption that the earth moves in a certain way, the sun is in a certain way at rest, the apparent irregularities with reference to the sun may be saved.' Philological considerations as well as the other notices which we possess about Heraclides make it practically certain that 'Heraclides of Pontus' is an interpolation and that Geminus said simply, 'a certain person', without any name, though doubtless meant Aristarchus of Samos.¹

Simplicius says that Alexander quoted this extract from the *epitome* of the ἐξήγησις by Geminus. As the original work was apparently made the subject of an abridgement, it is gathered that it must have been of considerable scope. It is a question whether ἐξήγησις means 'commentary' or 'exposition'; I am inclined to think that the latter interpretation is the correct one, and that Geminus reproduced Posidonius's work in its entirety with elucidations and comments. It seems to me to be suggested by the words added by Simplicius immediately after the extract 'this is the account given by Geminus, or Posidonius in Geminus, of the difference between physics and astronomy', which seems to imply that Geminus in our passage reproduced Posidonius textually.

'Introduction to the *Phaenomena*' attributed to Geminus

There remains the treatise, purporting to be by Geminus, which has come down to us under the title *Γεμίνου εἰσαγωγὴ εἰς τὰ Φαινόμενα*.² What, if any, is the relation of this work to the exposition of Posidonius's *Meteorologica* or the *epitome* of it just mentioned? One view is that the original *Isagoge* of Geminus and the ἐξήγησις of Posidonius were one and the same work, though the *Isagoge* as we have it is not by Geminus, but by an unknown compiler. The objections to this are, first, that it does not contain the extract given by Simplicius, which would have come in usefully at the beginning of an Introduction to Astronomy, nor the other extracts given by Alexander from Geminus and relating to the ra-

dly, that it does not anywhere mention the name of Posidonius (not, perhaps, an insuperable objection); and, secondly, that there are views expressed in it which are not held by Posidonius but contrary to them. Again, the compiler knows how to give a sound judgement as between conflicting views, writes in good style on the whole, and cannot have been the mere compiler of extracts from Posidonius which the view in question assumes him to be. It is in any case safer to assume that the *Isagoge* and the *Commentaries* were separate works. At the same time, the *Isagoge*, as we have it, contains errors which we cannot attribute to Posidonius. The choice, therefore, seems to lie between two alternatives: either the book is by Geminus in the main, but in the course of centuries suffered deterioration by interpolations, mistakes of copyists, and so on, or it is a compilation of extracts from an original *Isagoge* by Geminus with foreign inferior elements introduced either by the compiler himself or by other prentice hands. The result is a tolerable elementary treatise suitable for teaching purposes and containing the most important doctrines of Greek astronomy represented from the standpoint of Hipparchus. Chapter 1 treats of the earth, the solar year, the irregularity of the sun's motion, which is explained by the eccentric position of the sun's orbit relatively to the zodiac, the order and the periods of revolution of the planets and the moon. In § 23 we are told that all fixed stars do not lie on one spherical surface, but some are farther away than others—a doctrine due to the Stoics. Chapter 2, again, treats of the twelve signs of the zodiac, chapter 3 of the constellations, chapter 4 of the axis of the universe and the poles, chapter 5 of the circles on the sphere (the equator and the parallel circles, arctic, summer-tropical, winter-tropical, antarctic, the colure-circles, the zodiacal, the ecliptic, the horizon, the meridian, and the Milky Way), chapter 6 of Day and Night, their relative lengths in different latitudes, their lengthening and shortening, chapter 7 of the times which the twelve signs take to rise. Chapter 8

(76 years). Chapter 9 deals with the moon's phases, chapters 10, 11 with eclipses of the sun and moon, chapter 12 with the problem of accounting for the motions of the sun, moon, and planets, chapter 13 with Risings and Settings and the technical terms connected therewith, chapter 14 with the circles described by the fixed stars, chapters 15 and 16 with mathematical and physical geography, the zones, &c. (Chapter 16 follows Eratosthenes's evaluation of the circumference of the earth, not that of Posidonius). Chapter 17, on weather conditions, denies the popular theory that changes of atmospheric conditions depend on the rising and setting of certain stars and states that the predictions of weather (*ἐπισήματα*) in the calendars (*παραπήγματα*) are only derived from experience and observation, and have no scientific value. Chapter 18, on the *ἐξελιγμός*, the shortest period which contains an integral number of synodic months, of days, and of anomalistic months of the moon; this period is three times the Chinese period of 223 lunations used for predicting eclipses. Chapter 19 of the chapter deals with the maximum, mean, and minimum daily motion of the moon. The chapter as a whole does not correspond to the rest of the book; it deals with more general matters, and is thought by Manitius to be a fragment of a discussion to which the compiler did not feel himself bound. At the end of the work is a calendar (*Parapegma*) giving the number of days taken by the sun to traverse each sign of the zodiac, the risings and settings of various stars, and weather indications noted by various astronomers, Democritus, Eudoxus, Dositheus, Euctemon, Meton, Callippus; this calendar is unconnected with the rest of the book and the data are in several respects inconsistent with it, especially the division of the year into quarters which follows Callippus rather than Hipparchus. Hence it has been, since Heiberg's time, generally considered not to be the work of Geminus. Tittel, however, suggests that it is not impossible that Geminus may have reproduced an older *Parapegma* of Callippus.

XVI

SOME HANDBOOKS

THE description of the handbook on the elements of astronomy entitled the *Introduction to the Phaenomena* and ascribed to Geminus might properly have been reserved for this chapter. It was, however, convenient to deal with Geminus in close connexion with Posidonius; for Geminus has an exposition of Posidonius's *Meteorologica* related to the main work in such a way that Simplicius, in quoting a long passage from an epitome of this work, could attribute the passage to either Geminus or 'Posidonius in Geminus'; and it is evident that, in other subjects too, Geminus drew from, and was influenced by, Posidonius.

The small work *De motu circulari corporum caelestium* by PHILOMEDES (*Κλεομήδους κυκλική θεωρία*) in two Books is the work of a much less competent person, but is much more largely based on Posidonius. This is proved by several references to Posidonius by name, but it is specially true of the long first chapter of Book II (nearly half of the Book) which seems for the most part to be copied bodily from Posidonius, in accordance with the author's remark at the end of Book I that, in giving the refutation of the Epicurean opinion that the sun is just as large as it looks, namely one earth in diameter, he will give so much as suffices for such an introduction of the particular arguments used by 'certain philosophers who have written whole treatises on this one topic (the size of the sun), among whom is Posidonius'. The rest of the book then lies mainly in what is quoted from

works of Ptolemy, he can hardly, in any case, be later than the beginning of the second century A.D.

Book I begins with a chapter the object of which is to prove that the universe, which has the shape of a sphere, is limited and surrounded by void extending without limit in all directions, and to refute objections to this view. It then follows chapters on the five parallel circles in the heavens, the zones, habitable and uninhabitable (chap. 2); on the motion of the fixed stars and the independent (*πρὸς ἑαυτὰς*) movements of the planets including the sun and moon (chap. 3); on the zodiac and the effect of the sun's motion on it (chap. 4); on the inclination of the axis of the universe and its effects on the lengths of days and nights at different latitudes (chap. 5); on the inequality in the rate of increase of the lengths of the days and nights according to the time of the year, the different lengths of the seasons due to the motion of the sun in an eccentric circle, the difference between a sidereal night and an exact revolution of the universe owing to the separate motion of the sun (chap. 6); on the habitability of the globe including Britain and the 'island of Thule', which is said to have been visited by Pytheas, where, when the sun is in Cancer and visible, the day is a month long; and so on. Chap. 8 purports to prove that the universe is a sphere, by proving first that the earth is a sphere, and then that the moon is about it, and the ether about that, must necessarily be larger spheres. The earth is proved to be a sphere by the method of exclusion; it is assumed that the only possible shapes are that it is (a) flat and plane, or (b) hollow and spherical, or (c) square, or (d) pyramidal, or (e) spherical, and, the first four hypotheses being successively disposed of, only the last remains. Chap. 9 maintains that the earth is in the center of the universe; chap. 10, on the size of the earth, contains an interesting reproduction of the details of the measurements of the earth by Posidonius and Eratosthenes respectively, which have been given above in their proper places (p. 220, pp. 221-222). Chap. 11 argues that the earth is in the relation of a point to a sphere, i. e. is negligible in size in comparison with, the universe, even the sun's circle, but not the moon's circle (cf. p. 222).

Book II, chap. 1, is evidently the *pièce de résistance*.

g of an elaborate refutation of Epicurus and his followers, held that the sun is just as large as it *looks*, and further (according to Cleomedes) that the stars are lit up as rise and extinguished as they set. The chapter seems to be almost wholly taken from Posidonius; it ends with some of merely vulgar abuse, comparing Epicurus with Therapists with more of the same sort. The value of the chapter lies in certain historical traditions mentioned in it, and in the content of Posidonius's speculation as to the size and distance of the sun, which does, as a matter of fact, give results much nearer the truth than those obtained by Aristarchus, Hipparchus and Ptolemy. Cleomedes observes (1) that by means of gnomon-clocks it is found that the apparent diameter of the sun is $\frac{1}{750}$ th of the sun's circle, and that this method of measuring it is said to have been first invented by the Egyptians; (2) that Hipparchus is said to have found that the sun is 1,050 times the size of the earth, though, as regards this, we have the better authority of Adrastus (in Theon of Smyrna) and of Chalcidius, according to whom Hipparchus found the sun nearly 1,880 times the size of the earth (both of these refer of course to the solid content). We have already described Posidonius's method of arriving at the size and distance of the sun (pp. 220-1). After he has given this, Cleomedes, apparently deserting his guide, adds a calculation of his own relating to the sizes and distances of the moon and sun which shows how little he was capable of any scientific inquiry.¹ Chap. 2 purports to prove that the sun is

he says (pp. 146. 17-148. 27) that in an eclipse the breadth of the earth's shadow is stated to be two moon-breadths; hence, he says, it is credible (*πιθανόν*) that the earth is twice the size of the moon (this actually assumes that the breadth of the earth's shadow is equal to the diameter of the earth, or that the cone of the earth's shadow is under!). Since then the circumference of the earth, according to Eratosthenes, is 250,000 stades, and its diameter therefore 'more than 80,000' (he evidently takes $\pi = 3$), the diameter of the moon will be 40,000 stades. Now, the moon's circle being 750 times the moon's diameter, the radius of the moon's circle, i.e. the distance of the moon from the earth, will be $\frac{1}{3}$ th of this (i.e. $\pi = 3$) or 125 moon-diameters;

the size of the moon and the stars (chap. 3), the phases of the moon by the sun (chap. 4), the phases of the moon by its conjunctions with the sun (chap. 5), the eclipses of the moon (chap. 6), the maximum deviation in latitude of the planets (given as 5° for Venus, 4° for Mercury, 2° for Jupiter, 1° for Saturn), the maximum elongations of Mercury and Venus from the sun (20° and 50° respectively) and the synodic periods of the planets (Mercury 115 days, Venus 584 days, Mars 780 days, Jupiter 398 days, Saturn 378 days) (chap. 7).

There is only one other item of sufficient interest mentioned here. In Book II, chap. 6, Cleomedes tells us that there were stories of extraordinary eclipses of the moon more ancient of the mathematicians had vainly tried to explain'; the supposed 'paradoxical' case was that the moon while the sun seems to be still above the horizon, the moon rises in the east. The passage has been discussed (vol. i, pp. 6-7), where I have also shown that Cleomedes himself gives the true explanation of the phenomenon, namely that it is due to atmospheric refraction.

The first and second centuries of the Christian era saw a continuation of the work of writing manuals of arithmetic and tables to the different mathematical subjects. About the middle of the first century came NICOMACHUS, who wrote an *Introduction to Arithmetic* and an *Introduction to Harmony*; if we may judge from a remark of his own,¹ he would appear to have written an *Introduction to geometry* also. The *Arithmetical Introduction* has been sufficiently described above (vol. i, pp. 97-111).

There is yet another handbook which needs to be described separately, although we have had occasion to quote it several times already. This is the book by THEON of Smyrna, which goes by the title *Expositio rerum mathematicarum legendarum Platonem utilium*. There are two manuscripts of this work, contained in two Venice manuscripts respectively. The first was edited by Bullialdus (Paris, 1645); the second by T. H. Martin (Paris, 1849); the whole

¹ Nicom. *Arith.* ii. 6. 1.

d by E. Hiller (Teubner, 1878) and finally, with a French translation, by J. Dupuis (Paris, 1892).

Theon's date is approximately fixed by two considerations. It is clearly the person whom Theon of Alexandria called 'old Theon', τὸν παλαιὸν Θέωνα,¹ and there is no reason to doubt that he is the 'Theon the mathematician' (ὁ μαθηματικός) who is credited by Ptolemy with four observations of the planets Mercury and Venus made in A.D. 127, 129, 130 and 132.² The latest writers whom Theon himself mentions are Thrasyllus, who lived in the reign of Tiberius, and Apollonius the Peripatetic, who belongs to the middle of the second century A.D. Theon's work itself is a curious medley, valuable, not intrinsically, but for the numerous historical details which it contains. The title, which claims that the book contains things useful for the study of Plato, must not be taken too seriously. It was no doubt an elementary *introduction* or vade-mecum for students of philosophy, but it has little in it which has special reference to the mathematical questions raised in Plato. The connexion consists chiefly in the long proem quoting the views of Plato on the paramount importance of mathematics in the training of the philosopher, and the mutual relation of the five different sciences, arithmetic, geometry, stereometry, astronomy and music. The want of care shown by Theon in the quotations from particular dialogues of Plato prepares us for the patchwork character of the whole book.

In the first chapter he promises to give the mathematical elements most necessary for the student of Plato to know, arithmetic, music, and geometry, with its application to stereometry and astronomy.³ But the promise is by no means fulfilled as regards geometry and stereometry: indeed, in a later passage Theon seems to excuse himself from including analytical geometry in his plan, on the ground that all those who are likely to read his work or the writings of Plato may be assumed to have gone through an elementary course of analytical geometry.⁴ But he writes at length on figured

sidered that he was in this way sufficiently fulfilling his promise with regard to geometry and stereometry. (The geometrical definitions, of point, line, straight line, the dimensions, rectilinear plane and solid figures, especially parallelograms and parallelepipedal figures including *plinthides* (square bricks) and *δοκίδες* (beams), and figures with sides unequal every way (= *βωμίσκοι*), classification of solid numbers), are dragged in later (vol. 53-5 of the section on music)¹ in the middle of the discussion of proportions and means; if this passage is not an interpolation, it confirms the supposition that Theon included in his work only this limited amount of geometry and stereometry.

Section I is on Arithmetic in the same sense as Nicomachus' *Introduction*. At the beginning Theon observes that arithmetic will be followed by music. Of music in its various aspects, music in instruments (*ἐν ὀργάνοις*), music in numbers, i.e. musical intervals expressed in numbers or pure theoretical music, and the music or harmony in the universe, the instrumental kind (instrumental music) is not exactly essential, but the two must be discussed immediately after arithmetic.² The contents of the arithmetical section have been sufficiently indicated in the chapter on Pythagorean arithmetic (vol. i, pp. 11-12). It deals with the classification of numbers, odd, even, their subdivisions, prime numbers, composite numbers, equal or unequal factors, plane numbers subdivided into square, oblong, triangular and polygonal numbers, with their respective 'gnomons' and their properties as the successive terms of arithmetical progressions beginning with 1 as the first term, circular and spherical numbers, solid numbers with three factors, pyramidal numbers and triangular pyramidal numbers, perfect numbers with their correlates, the over-perfect and the deficient; this is practically what we find in Nicomachus. But the special value of Theon's exposition lies in the fact that it contains an account of the famous 'side-' and 'diameter-' numbers of the Pythagoreans.

¹ Theon of Smyrna, ed. Hiller, pp. 111-13. ² *Ib.*, pp. 16. 24-25.

³ *Ib.*, pp. 42. 10-45. 9. Cf. vol. i, pp. 91-3.

the Section on Music Theon says he will first speak of two kinds of music, the audible or instrumental, and the intelligible or theoretical subsisting in numbers, after which he promises to deal lastly with ratio as predicable of mathematical entities in general and the ratio constituting the harmony in the universe, 'not scrupling to set out once again things discovered by our predecessors, just as we have the things handed down in former times by the Pythagoreans, with a view to making them better known, without ourselves claiming to have discovered any of them'.¹ Then follows a discussion of audible music, the intervals which constitute the harmonies, &c., including substantial quotations from Thrasyllus and Adrastus, and references to views of Aristarchus, Hippasus, Archytas, Eudoxus and Plato. With chap. 17 (p. 72) begins the account of the 'harmony in the spheres', which turns into a general discussion of ratios, proportions and means, with more quotations from Plato, Eratosthenes and Thrasyllus, followed by Thrasyllus's *divisio numeris*, chaps. 35, 36 (pp. 87-93). After a promise to apply the latter division to the sphere of the universe, Theon attempts to return to the subject of proportion and means. This, however, does not occur till chap. 50 (p. 106), the intervening chapters being taken up with a discussion of *ἑκάς* and *τετρακτύς* (with eleven applications of the *ἑκάς*) and the mystic or curious properties of the numbers 1 to 10; here we have a part of the *theologumena* of Pythagoras. The discussion of proportions and the different kinds of means after Eratosthenes and Adrastus is again interrupted by the insertion of the geometrical definitions generally referred to (chaps. 53-5, pp. 111-13), after which Theon resumes the question of means for 'more precise' calculation.

The Section on Astronomy begins on p. 120 of Hiller's edition. Here again Theon is mainly dependent upon Ptolemy, from whom he makes long quotations. Thus, on the sphericity of the earth he says that for the neces-

comparison with the size of the whole, he quotes Eratosthenes and Dicaearchus as claiming to have discovered that the perpendicular height of the highest mountain above the level of the land is no more than 10 stades; and to obtain the diameter of the earth he uses Eratosthenes's figure of approximately 252,000 stades for the circumference of the earth, which, with the Archimedean value of $\frac{22}{7}$ for π , gives a diameter of about 80,182 stades. The principal astronomical circles in the heaven are next described (chaps. 5, 129-35); then (chap. 12) the assumed maximum deviations from the latitude are given, that of the sun being put at 1° , that of the moon and Venus at 12° , and those of the planets Mars, Jupiter and Saturn at 8° , 5° , 5° and 3° respectively. The obliquity of the ecliptic is given as the side of a regular polygon of 15 sides described in a circle, i.e. as 24° (chap. 23). Next the order of the orbits of the sun, moon and planets is explained (the system is of course geocentric); we are told that 'some of the Pythagoreans' made the order (reckoned outwards from the earth) to be moon, Mercury, Venus, Mars, Jupiter, Saturn, whereas (p. 142) Eratosthenes put the sun next to the moon, and the mathematicians, agreeing with Eratosthenes in this, differed only in the order in which they placed Venus and Mercury after the sun, some putting Mercury next and some Venus (p. 143). The order adopted by 'some of the Pythagoreans' is the Chaldaean order, which was followed by any Greek before Diogenes of Babylon (3rd century B.C.); 'some of the Pythagoreans' are therefore the later Pythagoreans (of whom Nicomachus was one); the other order, moon, sun, Venus, Mercury, Mars, Jupiter, Saturn, is that of Plato and the early Pythagoreans. In a footnote (p. 138 sq.) Theon quotes verses of Alexander 'the Aetolian' (not really the 'Aetolian', but Alexander of Ephesus, a contemporary of Cicero, or possibly Alexander of Miletus, whom Chalcidius calls him) assigning to each of the planets (including the earth, though stationary) with the sun and moon, the sphere of the fixed stars one note, the intervals between the notes being so arranged as to bring the nine notes of the octave, whereas with Eratosthenes and Plato the earth was excluded, and the eight notes of the octachord were

whole of this passage (chaps. 15 to 16, pp. 138-47) is no intended as the promised account of the 'harmony in universe', although at the very end of the work Theon states that this has still to be explained on the basis of Eudoxus's exposition combined with what he has already said himself.

The next chapters deal with the forward movements, the stationary points, and the retrogradations, as they respectively appear to us, of the five planets, and the 'saving of the phenomena' by the alternative hypotheses of eccentric circles and epicycles (chaps. 17-30, pp. 147-78). These hypotheses are compared, and the identity of the motion produced by the two is shown by Adrastus in the case of the sun (chaps. 26, 27, pp. 166-72). The proof is introduced with the interesting remark that 'Hipparchus says it is worthy of investigation by mathematicians why, on two hypotheses so different from one another, that of eccentric circles and that of concentric circles with epicycles, the same results appear to follow'. It is to be supposed that the proof of the identity could be no less than easy to a mathematician like Hipparchus; the remark perhaps merely suggests that the two hypotheses were considered quite independently, and it was not till later that the effect was discovered to be the same, when of course the identity would seem to be curious and a mathematical proof would immediately be sought. Another passage (p. 188) says that Hipparchus preferred the hypothesis of the epicycle, as being simpler. If this means that Hipparchus claimed to have discovered the epicycle-hypothesis, it must be a misapprehension, for Apollonius already understood the theory of epicycles in all its generality. According to Theon, the epicycle-hypothesis is more 'according to nature'; but it was presumably preferred because it was applicable to all the planets, whereas the eccentric-hypothesis, when originally suggested, applied only to the three superior planets; in order to make it apply to the inferior planets it is necessary to suppose the eccentricity described by the centre of the eccentric to be greater than the radius of the eccentric circle itself, which extension of the hypothesis, though known to Hipparchus, does not seem to have been referred to Apollonius.

We next have (chap. 31, p. 178) an allusion to the system of Eudoxus, Callippus and Aristotle, and a description (p. 180 sq.) of a system in which the 'carrying' spheres (called 'hollow') have between them 'solid spheres which by their own motion will roll (*ἀνελίσσουσι*) the carrying spheres in the opposite direction, being in contact with them'. These 'solid' spheres (which carry the planet fixed at a point on their surface) act in practically the same way as epicycles. In connexion with this description Theon (i.e. Adrastus) speaks (chap. 33, pp. 186-7) of two alternative hypotheses in which, by comparison with Chalcidius,¹ we recognize (after eliminating epicycles erroneously imported into both systems) the hypotheses of Plato and Heraclides respectively. It is this passage which enables us to conclude for certain that Heraclides made Venus and Mercury revolve in circles about the sun, like satellites, while the sun in its turn revolves in a circle about the earth as centre. Theon (p. 187) gives the maximum arcs separating Mercury and Venus respectively from the sun as 20° and 50° , these figures being the same as those given by Cleomedes.

The last chapters (chaps. 37-40), quoted from Adrastus, deal with conjunctions, transits, occultations and eclipses. The book concludes with a considerable extract from Dercyllides, a Platonist with Pythagorean leanings, who wrote (before the time of Tiberius and perhaps even before Varro) a book on Plato's philosophy. It is here (p. 198. 14) that we have the passage so often quoted from Eudemus:

'Eudemus relates in his Astronomy that it was Oenopides who first discovered the girdling of the zodiac and the revolution (or cycle) of the Great Year, that Thales was the first to discover the eclipse of the sun and the fact that the sun's period with respect to the solstices is not always the same, that Anaximander discovered that the earth is (suspended) on high and lies (substituting *κείται* for the reading of the manuscripts, *κινείται*, moves) about the centre of the universe, and that Anaximenes said that the moon has its light from the sun and (explained) how its eclipse

XVII

TRIGONOMETRY: HIPPARCHUS, MENELAUS, PTOLEMY

we have seen that *Sphaeric*, the geometry of the sphere, was very early studied, because it was required so soon as astronomy became mathematical; with the Pythagoreans the *Sphaeric*, applied to one of the subjects of the quadrivium, really meant astronomy. The subject was so far advanced by Euclid's time that there was in existence a regular book containing the principal propositions about great and small circles on the sphere, from which both Autolycus and Euclid quoted the propositions as generally known. These propositions, with others of purely astronomical interest, were collected afterwards in a work entitled *Sphaerica*, the *Three Books*, by THEODOSIUS.

Suidas has a notice, *s.v.* Θεόδωσιος, which evidently connects the author of the *Sphaerica* with another Theodosius, a Cypriot philosopher, since it calls him 'Theodosius, a philosopher, and attributes to him, besides the mathematical works, the *Stoic chapters*' and a commentary on the chapters of Theophrastus. Now the commentator on Theophrastus must have lived, at the earliest, to the second half of the second century A.D., whereas our Theodosius was earlier than Menelaus (about A.D. 100), who quotes him by name. The next notice by Suidas is of yet another Theodosius, a poet, who lived at Tripolis. Hence it was at one time supposed that the Theodosius was of Tripolis. But Vitruvius¹ mentions a Theodosius who invented a sundial 'for any climate': and

dosius was of Bithynia and not later in date than the first century (say 20 B.C.); but the order in which Strabo names makes it not unlikely that he was contemporary with Hipparchus, while the character of his *Sphaerica* is of a date even earlier rather than later.

Works by Theodosius.

Two other works of Theodosius besides the *Sphaerica*, namely *On habitations* and *On Days and Nights*, have been included in the 'Little Astronomy' (*μικρὴ ἀστρονομία*, sc. *τόπος*). These two treatises need no introduction. They are extant in Greek (in the great *Mathematica Graeca* 204 and others), but the Greek text has not yet been published. In the first, *On habitations*, 13 propositions, Theodosius explains the different phases of the sun to the daily rotation of the earth, and the parts of the whole system which are visible to inhabitants of different zones. In the second, *On Days and Nights*, 13 and 19 propositions in the two Books, Theodosius considers the arc of the ecliptic described by the sun each day, with a view to determining the course of the sun satisfied in order that the solstice may occur in a given place, and in order that the day and night may really be equal at the equinoxes; he shows that the variations in the day and night must recur at a certain time, if the length of the solar year be commensurable with that of the day, while on the contrary if not, they will not recur so exactly.

In addition to the works bearing on astronomy, Theodosius is said¹ to have written a commentary, now lost, on the *Method* of Archimedes (see above, pp. 27-34).

Contents of the *Sphaerica*.

We come now to the *Sphaerica*, which describes the geometry of the sphere from the point of view of this chapter. The book on the geometry of the sphere was wanted as a supplement to the *Elements* of Euclid. In the *Elements*

¹ Suidas, *loc. cit.*

the sphere except the theorem proved in XII. 16-18, that the volumes of two spheres are in the triplicate ratio of their diameters; apart from this, the sphere is only introduced in propositions about the regular solids, where it is proved that they are severally inscribable in a sphere, and it was doubtless with a view to his proofs of this property in each case that he gave a new definition of a sphere as the figure described by the revolution of a semicircle about its diameter, instead of the more usual definition (after the manner of the definition of a circle) as the locus of all points (in space instead of in a plane) which are equidistant from a fixed point (the centre). The doubt about the exclusion of the geometry of the sphere from the *Elements* was due to the fact that it was regarded as belonging to astronomy rather than pure geometry.

Proclus defines the sphere as 'a solid figure contained by a surface such that all the straight lines falling upon it from one point among those lying within the figure are equal in length to another', which is exactly Euclid's definition of a circle, 'solid' inserted before 'figure' and 'surface' substituted by 'line'. The early part of the work is then generally developed on the lines of Euclid's Book III on the circle. A plane section of a sphere is a circle (Prop. 1). The straight line from the centre of the sphere to the centre of any circular section is perpendicular to the plane of that section (Prop. 2; cf. 7, 23); thus a plane section serves for finding the centre of the sphere just as a chord does for finding that of a circle (Prop. 2). The propositions about tangent planes and the relation between the sizes of circular sections at different distances from the centre (5, 6) correspond to those of Books III. 16-19 and 15; as the small circle corresponds to a minor chord, the great circle ('greatest circle' in Greek) corresponds to the diameter. The poles of a circular section correspond to the extremities of the diameter bisecting the circle at right angles (Props. 8-10). Great circles bisecting one another (Props. 11-12) correspond to diameters which bisect one another (diameters), and great circles cutting small circles at right angles and passing through their poles (Props. 13-15) correspond to diameters bisecting small circles at right angles. The distance of any point of a great

circle from its pole is equal to the side of a square inscribed in the great circle and conversely (Props. 16, 17). Next come certain problems: To find a straight line equal to the diameter of any circular section or of the sphere itself (Props. 18, 19); to draw the great circle through any two given points on the surface (Prop. 20); to find the pole of any given circular section (Prop. 21). Prop. 22 applies Eucl. III. 3 to the sphere.

Book II begins with a definition of circles on a sphere which touch one another; this happens 'when the common section of the planes (of the circles) touches both circles'. Another series of propositions follows, corresponding to propositions in Eucl., Book III, for the circle. Parallel circular sections have the same poles, and conversely (Prop. 1, 2). Props. 3-5 relate to circles on the sphere touching one another and therefore having their poles on a great circle which also passes through the point of contact (Eucl. III. 11, [12] about circles touching one another). If a great circle touches a small circle, it also touches any other small circle equal and parallel to it (Props. 6, 7), and if a great circle be obliquely inclined to another circular section it touches each of two equal circles parallel to that section (Prop. 8). If two circles on a sphere cut one another, the great circle drawn through their poles bisects the intercepted segments of the circles (Prop. 9). If there are any number of parallel circles on a sphere, and any number of great circles drawn through their poles, the arcs of the parallel circles intercepted between any two of the great circles are similar, and the arcs of the great circles intercepted between any two of the parallel circles are equal (Prop. 10).

The last proposition forms a sort of transition to the pole of the treatise (II. 11-23 and Book III) which contains propositions of purely astronomical interest, though expressed in propositions in pure geometry without any specific reference to the various circles in the heavenly sphere. The propositions are long and complicated, and it would neither be

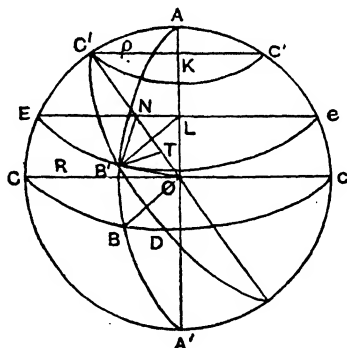
the stars which do not set, as seen by an observer at a particular place on the earth's surface; the pole of this is the pole in the heaven. A great circle which touches the pole and is obliquely inclined to the 'parallel circles' is the circle of the horizon; the parallel circles of course represent the apparent motion of the fixed stars in the diurnal rotation, with the pole of the heaven as pole. A second great circle obliquely inclined to the parallel circles is of course the circle of the zodiac or ecliptic. The greatest of the 'parallel circles' is naturally the equator. All that need be said of the preceding propositions (except two which will be mentioned separately) is that the sort of result proved is like that of propositions 12 and 13 of Euclid's *Phaenomena* to the effect that in the circle of the zodiac beginning with Cancer (or Capricorn) equal arcs set (or rise) in unequal times; those which are nearer the tropic circle take a longer time, those further from it a shorter; those which take the shortest time are adjacent to the equinoctial points; those which are equidistant from the equator rise and set in equal times. In like manner Theodosius (III. 8) in effect takes equal and corresponding arcs of the ecliptic all on one side of the equator, and through their extremities great circles touching the polar 'parallel' circle, and proves that the corresponding arcs of the equator intercepted between the latter great circles are unequal and that, of the said arcs, that corresponding to the arc of the ecliptic which is nearer the tropic circle is the greater. The successive great circles touching the polar circle are of course successive positions of the circle of the earth as the earth revolves about its axis, that is to say, the longer length of arc on the ecliptic takes a longer or shorter time to rise according as it is nearer to or farther from the equator. In other words, farther from or nearer to the equinoctial

points, however, obvious that investigations of this kind, which only prove that certain arcs are greater than others, and do not give the actual numerical ratios between them, are of no use for any practical purpose such as that of telling the length of the night by the stars, which was one of the fundamental problems in Greek astronomy; and in order to find

the required numerical ratios a new method had to be invented, namely trigonometry.

No actual trigonometry in Theodosius.

It is perhaps hardly correct to say that spherical trigonometry is nowhere referred to in Theodosius, for in III. 3 the congruence-theorem for spherical triangles corresponding to I. 4 is practically proved; but there is nothing in the work that can be called trigonometrical. The nearest approach is in III. 11, 12, where ratios between certain straight lines are compared with ratios between arcs. ACc (Prop. 11) is a great circle through the poles A, A' ; $CDc, C'D$ are two other great circles, both of which are at right angles to the plane of the equator, but CDc is perpendicular to AA' , while $C'D$ is inclined to it at an acute angle. Let any other great circle $AB'BA'$ be drawn



AA' cut CD in any point B between C and D , and $C'D$ in any point B' between C' and D' . Let the 'parallel' circle $EB'e$ be drawn through B' , and the 'parallel' circle $C'c'$ be the diameter of the 'parallel' circle touching the great circle $C'D$. Let L, K be the centres of the 'parallel' circles $EB'e$ and $C'c'$ respectively, and let R, ρ be the radii of the 'parallel' circles $EB'e$ and $C'c'$ respectively. It is required to prove that

$$2R : 2\rho > (\text{arc } CB) : (\text{arc } C'B').$$

measure NT along NO equal to NL , and join LB .

Then in the triangles $B'NT$, $B'NL$ two sides $B'N$, NT are equal to two sides $B'N$, NL , and the included angles (both right) are equal; therefore the triangles are equal in all respects, and $\angle NLB' = \angle NTB'$.

Now

$$\begin{aligned} 2R:2\rho &= OC':C'K \\ &= ON:NL \\ &= ON:NT \\ &[= \tan NTB':\tan NOB'] \\ &> \angle NTB':\angle NOB' \\ &> \angle NLB':\angle NOB' \\ &> \angle COB:\angle NOB' \\ &> (\text{arc } BC):(\text{arc } B'C'). \end{aligned}$$

a' , b' , c' are the sides of the spherical triangle $AB'C'$, this is equivalent (since the angle COB subtended by the arc BC is equal to A) to

$$\begin{aligned} 1:\sin b' &= \tan A:\tan a' \\ &> a:a', \end{aligned}$$

where $a = BC$, the side opposite A in the triangle ABC .

The proof is based on the fact (proved in Euclid's *Optics* and assumed as known by Aristarchus of Samos and Archimedes) that, if α , β are angles such that $\frac{1}{2}\pi > \alpha > \beta$, $\frac{\tan \alpha}{\tan \beta} > \alpha/\beta$.

While, therefore, Theodosius proves the equivalent of the formula, applicable in the solution of a spherical triangle right-angled at C , that $\tan \alpha = \sin b \tan A$, he is unable, for want of trigonometry, to find the actual value of α/a' , and can only find a limit for it. He is exactly in the same position as Aristarchus, who can only approximate to the values of the trigonometrical ratios which he needs, e.g. $\sin 1^\circ$, $\cos 1^\circ$, $\sin 3^\circ$, by bounding them within upper and lower limits with the aid of the inequalities

$$\frac{\tan \alpha}{\tan \beta} > \frac{\alpha}{\beta} > \frac{\sin \alpha}{\sin \beta},$$

where $\frac{1}{2}\pi > \alpha > \beta$.

corresponding proposition in Menelaus's *Sphaerica* dealing with the more general case in which C' , instead of being the tropical point on the ecliptic, is, like B' , any point between the tropical point and D . If R, ρ have the same meaning as above and r_1, r_2 are the radii of the parallels through B' and the new C' , Menelaus proves that

$$\frac{\sin a}{\sin a'} = \frac{R\rho}{r_1 r_2},$$

which, of course, with the aid of Tables, gives the means of finding the actual values of a or a' when the other elements are given.

The proposition III. 12 of Theodosius proves a result similar to that of III. 11 for the case where the great circles $AC'C$, instead of being great circles through the poles, are great circles touching 'the circle of the always-visible stars' i.e. different positions of the horizon, and the points C and C' are any points on the arc of the oblique circle between the poles and the equinoctial points; in this case, with the same notation, $4R:2\rho > (\text{arc } BC):(\text{arc } B'C')$.

It is evident that Theodosius was simply a laborious compiler, and that there was practically nothing original in his work. It has been proved, by means of propositions proved *verbatim* or assumed as known by Autolycus in his *Sphaera* and by Euclid in his *Phaenomena*, that the first five propositions in Theodosius are pre-Euclidean, I. 1, 6 a, 12, 13, 15, 20; II. 1, 2, 3, 5, 8, 9, 10 a, 13, 15, 17, 18, 19, 20. III. 1 b, 2, 3, 7, 8, those shown in thick type being original word for word.

The beginnings of trigonometry.

But this is not all. In Menelaus's *Sphaerica*, III. 11, there is a reference to the proposition (III. 11) of Theodosius proved above, and in Gherard of Cremona's translation from the Arabic, as well as in Halley's translation from the Latin of Jacob b. Machir, there is an addition to the effect that the proposition was used by Apollonius in a book the title of which is given in the two translations in the alt

s 'liber aggregativus' and 'liber de principiis universalibus'. Each of these expressions may well mean the work of Apollonius which Marinus refers to as the 'General Treatise' (ἡ καθόλου πραγματεία). There is no apparent reason to doubt that the remark in question was really contained in Menelaus's original work; and, even if it is an Arabian interpolation, it is not likely to have been made without some definite authority. If then Apollonius was the discoverer of the proposition, the fact affords some ground for thinking that the beginnings of trigonometry go as far back, at least, as Apollonius. Tannery¹ indeed suggested that not Apollonius but Archimedes before him may have composed a 'table of chords', or at least shown the way to such compilation, Archimedes in the work of which we possess a fragment in the *Measurement of a Circle*, and Apollonius in the *ὠκυτόκιον*, where he gave an approximation to the value of π closer than that obtained by Archimedes; Tannery compares the Indian Table of Sines in the *Sūrya-Siddhānta*, where the angles go by 24ths of a right angle ($1/24\text{th} = 3^\circ 45'$, $2/24\text{th} = 7^\circ 30'$, &c.), as possibly showing Greek influence. This is, however, in the region of conjecture; the first person to make systematic use of trigonometry is, so far as we know, Hipparchus.

HIPPARCHUS, the greatest astronomer of antiquity, was born at Nicaea in Bithynia. The period of his activity is indicated by references in Ptolemy to observations made by Hipparchus within the limits of which are from 161 B.C. to 126 B.C. Ptolemy further says that from Hipparchus's time to the beginning of the reign of Antoninus Pius (A.D. 138) was 265 years.² The most important observations made by Hipparchus were made at Rhodes, though an observation of the vernal equinox at Alexandria on March 24, 146 B.C., recorded by him is probably his own. His main contributions to theoretical and practical astronomy can here only be indicated in the briefest manner.

¹ Tannery, *Recherches sur l'hist. de l'astronomie ancienne*, p. 64.
² Ptolemy, *Syntaxis*, vii. 2 (vol. ii, p. 15).

Discovery of precession.

1. The greatest is perhaps his discovery of the of the equinoxes. Hipparchus found that the *Spica* was, at the time of his observation of it, from the autumnal equinoctial point, whereas he deduced from observations recorded by Timocharis that Timocharis made the distance 8° . Consequently the motion had advanced to 2° in the period between Timocharis's observation, 283 or 295 B.C., and $129\frac{7}{8}$ B.C., a period, that is, of 166 years; this gives about $46.8''$ or $43.4''$ a year, a value with the true value of $50.3757''$.

Calculation of mean lunar month.

2. The same discovery is presupposed in his work on the *length of the Year*, in which, by comparing an observation of the summer solstice by Aristarchus in $281\frac{1}{10}$ B.C. with one made by him in $136\frac{5}{5}$ B.C., he found that after 145 years (the interval between the two dates) the summer solstice occurred a day-and-night earlier than it should on the assumption of a year exactly $365\frac{1}{4}$ days to the year; hence he concluded that the tropical year contained about $\frac{1}{360}$ th of a day-and-night less than $365\frac{1}{4}$ days. This agrees very nearly with Callippus's statement that Hipparchus's cycle was 304 years, or 111,040 days, instead of 111,036 ($=27,759 \times 4$). Counting in the 12 \times 304 + 112 (intercalary) months, or 3,760 months, Hipparchus made the mean lunar month 29 days 44 min. $2\frac{1}{2}$ sec., which is less than a second out in comparison with the present accepted figure of 29.53059 days!

3. Hipparchus attempted a new determination of the motion by means of exact equinoctial and solstitial observations; he reckoned the eccentricity of the sun's orbit and fixed the apogee at the point $5^{\circ}30'$ of *Gemini*. His most remarkable still was his investigation of the precession of the equinoxes. He determined the eccentricity and the position of the orbit to the ecliptic, and by means of observations of eclipses determined the moon's position with extraordinary accuracy (as remarked above). We

the lengths of the mean synodic, the sidereal, the analeptic and the draconitic month obtained by Hipparchus exactly with Babylonian cuneiform tables of date not later than Hipparchus, and it is clear that Hipparchus was in possession of all the results established by Babylonian astronomy.

Improved estimates of sizes and distances of sun and moon.

Hipparchus improved on Aristarchus's calculations of the sizes and distances of the sun and moon, determining the true diameters more exactly and noting the changes in them; he made the mean distance of the sun $1,245 D$, the mean distance of the moon $33\frac{2}{3} D$, the diameters of the sun and moon $12\frac{1}{3} D$ and $\frac{1}{3} D$ respectively, where D is the mean diameter of the earth.

Epicycles and eccentrics.

Hipparchus, in investigating the motions of the sun, moon and planets, proceeded on the alternative hypotheses of epicycles and eccentrics; he did not invent these hypotheses, which were already fully understood and discussed by Ptolemy. While the motions of the sun and moon could not be accounted for by the simple epicycle and eccentric hypotheses, Hipparchus found that for the planets it was necessary to combine the two, i.e. to superadd epicycles to motion in eccentric circles.

Catalogue of stars.

Hipparchus compiled a catalogue of fixed stars including 850 or more such stars; apparently he was the first to state their positions in terms of coordinates in relation to the ecliptic (latitude and longitude), and his table distinguished the apparent sizes of the stars. His work was continued by Ptolemy, who produced a catalogue of 1,022 stars which, owing to an error in his solar tables affecting all his longi-

7. He made great improvements in the instruments of observations. Among those which he used were an improved dioptra, a 'meridian-instrument' designed for observing the meridian only, and a universal instrument (*ἀστρολάβος*) for more general use. He also made a globe on which he showed the positions of the fixed stars as determined by him; it appears that he showed a larger number of stars on his globe than in his catalogue.

Geography.

In geography Hipparchus wrote a criticism of Eratosthenes in great part unfair. He checked Eratosthenes's measurements by means of a sort of triangulation; he insisted on the necessity of applying astronomy to geography, of fixing the positions of places by latitude and longitude, and of determining longitudes by observations of lunar eclipses.

Outside the domain of astronomy and geography, Hipparchus wrote a book *On things borne down by their weight*, in which Simplicius (on Aristotle's *De caelo*, p. 264 sq.) mentions two propositions. It is possible, however, that even in this work Hipparchus may have applied his doctrine to the motions of the heavenly bodies.

In pure mathematics he is said to have considered a problem in permutations and combinations, the problem of finding the number of different possible combinations of 10 axioms or assumptions, which he made to be 103,049 (*v.l.* 310,952 according as the axioms were affirmed or denied); it seems impossible to make anything of these figures. The *Fihrist* attributes to him works 'On the art of permutations' known by the title of the Rules' and 'On the division of numbers', we have no confirmation: Suter suspects some confusion in view of the fact that the article immediately following the *Fihrist* is on Diophantus, who also 'wrote on the theory of algebra'.

¹ Plutarch, *Quaest. Conviv.* viii. 9. 3, 732 F, *De Stoicorum rebus* 1047 D.

First systematic use of Trigonometry.

come now to what is the most important from the point of view of this work, Hipparchus's share in the development of trigonometry. Even if he did not invent it, Hipparchus is the first person of whose systematic use of trigonometry we have documentary evidence. (1) Theon of Alexandria says on the *Syntaxis* of Ptolemy, à propos of Ptolemy's Table of Chords in a circle (equivalent to sines), that Hipparchus, too, wrote a treatise in twelve books on the calculation of lines (i.e. chords) in a circle, while another in six books was written by Menelaus.¹ In the *Syntaxis* I. 10 Ptolemy gives the necessary explanations as to the notation used in his Table. The circumference of the circle is divided into 360 parts or degrees; the diameter is also divided into 360 parts, and one of such parts is the unit of length in terms of which the length of each chord is expressed; each part, whether of the circumference or diameter, is divided into 60 parts, each of these again into 60, and so on, according to the method of sexagesimal fractions. Ptolemy then sets out the whole number of propositions in plane geometry upon which the calculation of the chords in the Table is based (*διὰ τῶν γραμμῶν μεθοδικῆς αὐτῶν συστάσεως*). The propositions are famous, and it cannot be doubted that Hipparchus used a set of propositions of the same kind, though his work probably ran to much greater length. As Ptolemy merely set himself to give the necessary propositions in the simplest form possible, it will be better to give them under Ptolemy rather than here. (2) Pappus, in speaking of Euclid's propositions about the inequality of the times which equal arcs of the zodiac take to rise, observes that 'Hipparchus in his book *On the rising of the twelve signs of the zodiac* shows by means of numerical calculations (*δι' ἀριθμῶν*) that equal arcs of the circle beginning with Cancer which set in times having a certain relation to one another do not everywhere show the

in relation to one another; they could not calculate the times. As Hipparchus proved corresponding propositions by means of *numbers*, we can only conclude that he used propositions in spherical trigonometry, calculating arcs from which are given, by means of tables. (3) In the only one of his which survives, the *Commentary on the Phaenomena of Eudoxus and Aratus* (an early work anterior to the discovery of the precession of the equinoxes), Hipparchus states that (presumably in the latitude of Rhodes) a star which lies $27\frac{1}{3}^{\circ}$ north of the equator describes above the horizon arcs containing 3 minutes less than $15/24$ ths of the circumference¹; then, after some more inferences, he says, 'For the truth of the aforesaid facts is proved by means of lines (*διὰ γραμμῶν*) in the general treatises on these matters composed by me'. In other places² of the *Commentary* he alludes to a work *On simultaneous risings* (*τὰ περὶ τῶν συνανατολῶν*) and in II. 4. 2 he says he will state summarily, about the risings of the fixed stars, along with what sign of the zodiac it rises, the sets and from which degree to which degree of each sign it rises or sets in the regions about Greece or wherever the longest day is $14\frac{1}{2}$ equinoctial hours, adding that he has special proofs in another work designed so that it is possible to find in practically every place in the inhabited earth to what hour the differences between the concurrent risings and sets occur. Where Hipparchus speaks of proofs 'by means of lines' (*διὰ γραμμῶν*) does not mean a merely graphical method, by construction only, but theoretical determination by geometry, followed by numerical calculation, just as Ptolemy uses the expression *ἐκ τῶν γραμμῶν* of his calculation of chords and the expressions *σφαιρικά*, *δείξεις* and *γραμμικαὶδείξεις* of the fundamental propositions in spherical trigonometry (Menelaus's theorem applied to the sphere) and its various applications to particular cases. It is significant that in the *Syntaxis* VIII. 5, where Ptolemy applies the proposition to the very problem of finding the times of concurrent rising, culmination and setting of fixed stars, he says that the times can be obtained 'by means of lines' (*διὰ γραμμῶν*).

Commentary, he used the formulae of spherical trigonometry to get his results. In the particular case where it is required to find the time in which a star of $27\frac{1}{3}^\circ$ northern declination describes, in the latitude of Rhodes, the portion of its path above the horizon, Hipparchus must have used the equivalent of the formula in the solution of a right-angled spherical triangle, $\tan b = \cos A \tan c$, where C is the right angle. Whether, like Ptolemy, Hipparchus obtained the formulae, such as this one, which he used from different applications of the one general theorem (Menelaus's theorem) is not possible to say. There was of course no difficulty in calculating the tangent or other trigonometrical function of an angle if only a table of sines was given; for Hipparchus and Ptolemy were both aware of the fact expressed by $\sin^2 \alpha + \cos^2 \alpha = 1$ or, as they would have written it,

$$(\text{crd. } 2\alpha)^2 + \{\text{crd. } (180^\circ - 2\alpha)\}^2 = 4r^2,$$

where $\text{crd. } 2\alpha$ means the chord subtending an arc 2α , and r the radius, of the circle of reference.

Table of Chords.

We have no details of Hipparchus's Table of Chords sufficient to enable us to compare it with Ptolemy's, which goes in half-degrees, beginning with angles of $\frac{1}{2}^\circ$, 1° , $1\frac{1}{2}^\circ$, and so on. But Heron¹ in his *Metrica* says that 'it is proved in the case about chords in a circle' that, if a_9 and a_{11} are the sides of a regular enneagon (9-sided figure) and hendecagon (11-sided figure) inscribed in a circle of diameter d , then (1) $a_9 = \frac{1}{3}d$, $a_{11} = \frac{2}{5}d$ very nearly, which means that $\sin 20^\circ$ was taken as equal to 0.3333... (Ptolemy's table makes it $0 + \frac{31}{60} + \frac{16\frac{1}{2}}{60^2}$); so that the first approximation is $\frac{1}{3}$), and $\sin 180^\circ$ or $\sin 16^\circ 21' 49''$ was made equal to 0.28 (this corresponds to the chord subtending an angle of $32^\circ 43' 38''$, nearly the mean between $32\frac{1}{2}^\circ$ and 33° and the mean between the two

by $\frac{1}{800}$ from $\frac{168}{800}$ or $\frac{7}{25}$, Heron's figure). There is little doubt that it is to Hipparchus's work that Heron refers, though the author is not mentioned.

While for our knowledge of Hipparchus's trigonometry we have to rely for the most part upon what we can infer from Ptolemy, we fortunately possess an original source of information about Greek trigonometry in its highest development in the *Sphaerica* of Menelaus.

The date of MENELAUS of Alexandria is roughly indicated by the fact that Ptolemy quotes an observation of his made in the first year of Trajan's reign (A.D. 98). He was therefore a contemporary of Plutarch, who in his *Life of Trajan* represents him as being present at the dialogue *De facili orbe lunae*, where (chap. 17) Lucius apologizes to Menelaus 'mathematician' for questioning the fundamental proposition in optics that the angles of incidence and reflection are equal.

He wrote a variety of treatises other than the *Sphaerica*. We have seen that Theon mentions his work on *Chords of a Circle* in six Books. Pappus says that he wrote a treatise (*πραγματεία*) on the setting (or perhaps only rising) of the sun at different arcs of the zodiac.¹ Proclus quotes an alternative proof by him of Eucl. I. 25, which is direct instead of *reductio ad absurdum*,² and he would seem to have avoided the latter kind of proof throughout. Again, Pappus, speaking of the many complicated curves 'discovered by Demetrius of Alexandria (in his "Linear considerations") and by Phylas of Tyana as the result of interweaving plectoids and conic surfaces of all kinds', says that one curve in particular was investigated by Menelaus and called by him 'paradoxical' (*παράδοξος*);³ the nature of this curve can only be conjectured (see below).

But Arabian tradition refers to other works by Menelaus: (1) *Elements of Geometry*, edited by Thābit b. Qurra, in three Books, (2) a Book on triangles, and (3) a work the title of which is translated by Wenrich *de cognitione quantitatis discretæ corporum permixtorum*. Light is thrown on the last title by the fact that it is the same as the title of a work by Pappus.

about the hydrostatic balance, i.e. about the determination of the specific gravity of homogeneous or mixed in the course of which he mentions Archimedes and others (among others) as authorities on the subject; hence it is probable that (3) must have been a book on hydrostatics discussing such problems as that of the crown solved by Archimedes. The alternative proof of Eucl. I. 25 quoted by Menelaus might have come either from the *Elements of Geometry* or from a Book on triangles. With regard to the geometry, the *Kitāb al-Musarrafāt* (written by three sons of Mūsā b. Shākir in the ninth century) says that it contained a solution of the problem of the cube, which is none other than that of Archimedes. The solution of Archytas having employed the intersection of a torus and a cylinder (with a cone as well), it would, on the assumption that Menelaus reproduced the solution, be a certain appropriateness in the suggestion of Menelaus that the curve which he called the *παράδοξος* was in reality the curve of double curvature, known by the name of Viviani, which is the intersection of a sphere and a cylinder touching it internally and having for its radius the radius of the sphere. This curve is a particular case of Eudoxus's *hippopede*, and it has the property that the area left outside the curve of the surface of the hemisphere which it lies on is equal to the square on the diameter of the sphere; the fact of the said area being squareable would justify the application of the word *παράδοξος* to the curve, and the quadrature itself would not probably be beyond the power of the Greek mathematicians, as witness Pappus's solution of the area cut off between a complete turn of a spiral on a sphere and the great circle touching it at a point.²

The *Sphaerica* of Menelaus.

The treatise in three Books is fortunately preserved in Arabic, and although the extant versions differ con-

edition (Oxford, 1758). The former is unserviceable. Maurolycus's manuscript was very imperfect, and trying to correct and restore the propositions, several of his own. Halley seems to have made a translation of the Hebrew version of the work by Machir (about 1273), although he consulted Arabic manuscripts to some extent, following them, e.g., in dividing the three Books instead of two. But an earlier version from the Arabic is available in manuscripts of the thirteenth to fifteenth centuries at Paris and elsewhere; this without doubt that made by the famous translator of Cremona (1114-87). With the help of Halley's, Gherard's translation, and a Leyden manuscript of the redaction of the work by Abū-Nasr-Manṣūr (A.D. 1007-8, Björnbo has succeeded in presenting an accurate reproduction of the contents of the *Sphaerica*.¹

Book I.

In this Book for the first time we have the concise definition of a *spherical triangle*. Menelaus does not give the usual definitions of points and circles on the sphere, e.g. pole, great circle, small circle, but begins with that of a spherical triangle as 'the area included by three great circles on the surface of a sphere', subject to the condition (Def. 2) that each of the sides or legs of the triangle be an arc less than a semicircle. The angles of the triangle are defined as angles contained by the arcs of great circles on the sphere (Def. 3), and one such angle is equal to or greater than a right angle according as the planes containing the arcs forming the angle are inclined at the same angle as, or a greater angle than, the planes of the arcs forming the other angle (Def. 4). The angle is a right angle if the planes of the arcs are perpendicular to each other (Def. 5). The angles (Def. 6). Pappus tells us that Menelaus in his definition calls the figure in question (the spherical triangle) 'a triangle' (*τρίπλευρον*)²; the word *triangle* (*τρίγωνον*) was

¹ Björnbo, *Studien über Menelaos' Sphärik* (Abhandlungen der math. Wissenschaften, Heft xiv. 1902).

² Pappus, vi, p. 476. 16.

already appropriated for the plane triangle. We should gather from this, as well as from the restriction of the definitions to the spherical triangle and its parts, that the discussion of the spherical triangle as such was probably new; and if the preface in the Arabic version addressed to a prince and beginning with the words, 'O prince! I have discovered an excellent method of proof...' is genuine, we have confirmatory evidence in the writer's own claim.

Menelaus's object, so far as Book I is concerned, seems to have been to give the main propositions about spherical triangles corresponding to Euclid's propositions about plane triangles. At the same time he does not restrict himself to Euclid's methods of proof even where they could be adapted to the case of the sphere; he avoids the form of proof by *reductio ad absurdum*, but, subject to this, he prefers the easiest proofs. In some respects his treatment is more complete than Euclid's treatment of the analogous plane cases. In the congruence-theorems, for example, we have I. 4a corresponding to Eucl. I. 4, I. 4b to Eucl. I. 8, I. 14, 16 to Eucl. I. 26a, b; but Menelaus includes (I. 13) what we know as the 'ambiguous case', which is enunciated on the lines of Eucl. VI. 7. I. 12 is a particular case of I. 16. Menelaus includes also the further case which has no analogue in plane triangles, that in which the three angles of one triangle are severally equal to the three angles of the other (I. 17). He makes, moreover, no distinction between the congruent and the symmetrical, regarding both as covered by congruent. I. 1 is a problem, to construct a spherical angle equal to a given spherical angle, introduced only as a lemma because required in later propositions. I. 2, 3 are the propositions about isosceles triangles corresponding to Eucl. I. 5, 6; Eucl. I. 18, 19 (greater side opposite greater angle and vice versa) have their analogues in I. 7, 9, and Eucl. I. 24, 25 (two sides respectively equal and included angle, or third side, in one triangle greater than included angle, or third side, in the other) in I. 8. I. 5

Eucl. I. 16, 32 are not true of spherical triangles, and Menelaus has therefore the corresponding but different propositions. I. 10 proves that, with the usual notation a, b, c for the sides and opposite angles of a spherical triangle, the exterior angle at C , or $180^\circ - C$, $< =$ or $>$ according as $c + a > =$ or $< 180^\circ$, and vice versa. The proof of this and the next proposition shall be given as specimen.

In the triangle ABC suppose that $c + a > =$ or $< 180^\circ$; D be the pole opposite to A .

Then, according as $c + a > =$ or $< 180^\circ$, $BC > =$ or $<$ (since $AD = 180^\circ$),

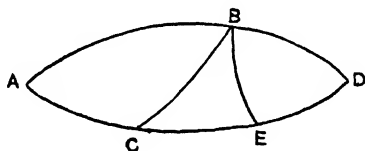
and therefore $\angle D > =$ or $< \angle BCD (= 180^\circ - C)$, [I.]

i.e. (since $\angle D = \angle A$) $180^\circ - C < =$ or $> A$.

Menelaus takes the converse for granted.

As a consequence of this, I. 11 proves that $A + B + C > 180^\circ$.

Take the same triangle ABC , with the pole D opposite



to A , and from B draw the great circle BE such that $\angle DBE = \angle BDE$.

Then $CE + EB = CD < 180^\circ$, so that, by the preceding proposition, the exterior angle ACB to the triangle BCD is greater than $\angle CBE$,

i.e. $C > \angle CBE$.

Add A or $D (= \angle EBD)$ to the unequals;

therefore $C + A > \angle CBD$,

whence $A + B + C > \angle CBD + B$ or 180° .

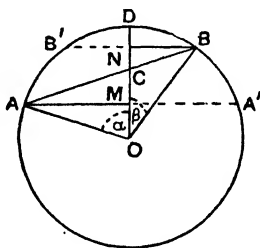
After two lemmas I. 21, 22 we have some propositions in

the triangle ABC with great circles drawn through B to AC (between A and C) in D , E respectively, and the where D and E coincide, and they prove different results from different relations between a and c ($a > c$), combined with the equality of AD and EC (or DC), of the angles and EBC (or DBC), or of $a + c$ and $BD + BE$ (or $2BD$) respectively, according as $a + c < =$ or $> 180^\circ$.

Book II has practically no interest for us. The object of it is to establish certain propositions, of astronomical interest which are nothing more than generalizations or extensions of propositions in Theodosius's *Sphaerica*, Book III. Theodosius III. 5, 6, 9 are included in Menelaus II. 10, Theodosius III. 7-8 in Menelaus II. 12, while Menelaus II. 11 is an extension of Theodosius III. 13. The proofs are quite different from those of Theodosius, which are generally very convoluted.

Book III. Trigonometry.

As will have been noticed that, while Book I of Menelaus is devoted to the geometry of the spherical triangle, neither Book I nor Book II contains any trigonometry. This is reserved for Book III. As I shall throughout express the various results obtained in terms of the trigonometrical ratios, sine, cosine, tangent, it is necessary to explain once for all that the Greeks did not use this terminology, but, instead of sines, they used chords subtended by arcs of a circle. In the accompanying figure let the arc AD of a circle subtend an angle α at the centre O . Draw AM perpendicular to OD , and produce it to meet the circle again in A' . Then $\sin \alpha = AM/AO$, and AM is $\frac{1}{2}AA'$, half the chord subtended by an arc of 2α at the centre, which may usually be denoted by $\frac{1}{2}(\text{crd. } 2\alpha)$.



(α) 'Menelaus's theorem' for the sphere.

The first proposition of Book III is the famous 'Menelaus's theorem' with reference to a spherical triangle and any transversal (great circle) cutting the sides of a triangle, produced if necessary. Menelaus does not, however, use a spherical triangle in his enunciation, but enunciates the proposition in terms of intersecting great circles. 'Between two arcs AEC of great circles are two other arcs of great circles ADF and BFE which intersect them and also intersect each other in F . All the arcs are less than a semicircle. It is required to prove that

$$\frac{\sin CE}{\sin EA} = \frac{\sin CF}{\sin FD} \cdot \frac{\sin DB}{\sin BA}.$$

It appears that Menelaus gave three or four cases, sufficient to prove the theorem completely. The proof depends on simple propositions which Menelaus assumes without proof; the proof of them is given by Ptolemy.

(1) In the figure on the last page, if OD be a radius cutting a chord AB in C , then

$$AC:CB = \sin AD:\sin DB.$$

For draw AM , BN perpendicular to OD . Then

$$\begin{aligned} AC:CB &= AM:BN \\ &= \frac{1}{2}(\text{crd. } 2AD) : \frac{1}{2}(\text{crd. } 2DB) \\ &= \sin AD:\sin DB. \end{aligned}$$

(2) If AB meet the radius OC produced in T , then

$$AT:BT = \sin AC:\sin BC.$$



$$AT:TB = AM:BN$$

$$= \frac{1}{2} (\text{crd. } 2 AC) : \frac{1}{2} (\text{crd. } 2 BC)$$

$$= \sin AC : \sin BC.$$

Now let the arcs of great circles ADB , AEC be cut by the arcs of great circles DFC , BFE which themselves meet in F .

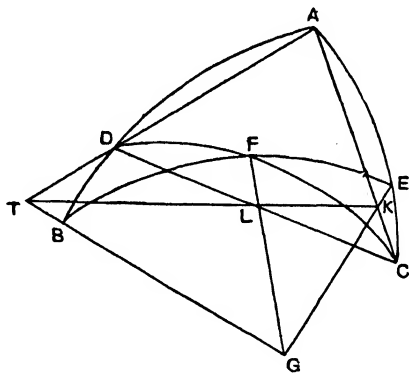
Let G be the centre of the sphere and join GB , GF , GE , AD .

Then the straight lines AD , GB , being in one plane, are either parallel or not parallel. If they are not parallel, they will meet either in the direction of D , B or of A , G .

Let AD , GB meet in T .

Draw the straight lines AKC , DLC meeting GE , GF in K , L respectively.

Then K , L , T must lie on a straight line, namely the straight line which is the section of the planes determined by the arc EFB and by the triangle ACD .¹



Thus we have two straight lines AC , AT cut by the two straight lines CD , TK which themselves intersect in L .

Therefore, by Menelaus's proposition in plane geometry,

$$\frac{CK}{KA} = \frac{CL}{LD} \cdot \frac{DT}{TA}.$$

¹ So Ptolemy. In other words, since the straight lines GB , GE , GF , which are in one plane, respectively intersect the straight lines AD , AC , CD which are also in one plane, the points of intersection T , K , L are in both planes, and therefore lie on the straight line in which the planes intersect.

But, by the propositions proved above,

$$\frac{CK}{KA} = \frac{\sin CE}{\sin EA}, \quad \frac{CL}{LD} = \frac{\sin CF}{\sin FD}, \quad \text{and} \quad \frac{DT}{TA} =$$

therefore, by substitution, we have

$$\frac{\sin CE}{\sin EA} = \frac{\sin CF}{\sin FD} \cdot \frac{\sin DB}{\sin BA}.$$

Menelaus apparently also gave the proof for which AD , GB meet towards A , G , and in which parallel respectively, and also proved that in like the above figure,

$$\frac{\sin CA}{\sin AE} = \frac{\sin CD}{\sin DF} \cdot \frac{\sin FB}{\sin BE}$$

(the triangle cut by the transversal being here $CHADC$). Ptolemy¹ gives the proof of the above and dismisses the last-mentioned result with a 'simila

(β) *Deductions from Menelaus's Theorem*

III. 2 proves, by means of I. 14, 10 and III. 1, that if $A'B'C'$ be two spherical triangles in which $A = A'$ and $B = B'$ are either equal or supplementary, $\sin c / \sin a = \sin c' / \sin a'$ and conversely. The particular case in which $C = C'$ angles gives what was afterwards known as 'quattuor quantatum' and was fundamental in spherical trigonometry.² A similar association attaches to III. 3, which is the so-called 'tangent' or 'shadow' theorem of the Arabs. If ABC , $A'B'C'$ be triangles right-angled at C , C' are equal and both either $>$ or $< 90^\circ$, and A , A' be the poles of AC , $A'C'$, then

$$\frac{\sin AB}{\sin AC} = \frac{\sin A'B'}{\sin A'C'} \cdot \frac{\sin BP}{\sin B'P'}.$$

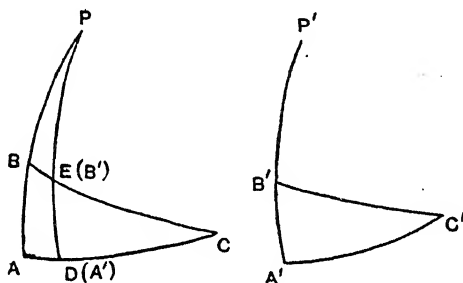
Apply the triangles so that C' falls on C , $C'B'$ on CB and $C'A'$ on CA as CD ; then the result follows from III. 1. Since $\sin BP = \cos AB$, and $\sin B'P' = \cos A'B'$, the result becomes

$$\sin CA \cdot \tan AB$$

It follows at once (Prop. 4) that, if $AM, A'M'$ are great circles drawn perpendicular to the bases $BC, B'C'$ of two spherical triangles $ABC, A'B'C'$ in which $B = B', C = C'$,

$$\frac{\sin BM}{\sin B'M'} = \frac{\sin MC}{\sin M'C'} \quad \left(\text{since both are equal to } \frac{\tan AM}{\tan A'M'} \right).$$

III. 5 proves that, if there are two spherical triangles $ABC,$



$A'B'C'$ right-angled at A, A' and such that $C = C'$, while b and b' are less than 90° ,

$$\frac{\sin(a+b)}{\sin(a-b)} = \frac{\sin(a'+b')}{\sin(a'-b')},$$

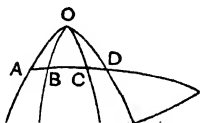
from which we may deduce¹ the formula

$$\frac{\sin(a+b)}{\sin(a-b)} = \frac{1 + \cos C}{1 - \cos C},$$

which is equivalent to $\tan b = \tan a \cos C$.

(γ) *Anharmonic property of four great circles through one point.*

But more important than the above result is the fact that the proof assumes as known the anharmonic property of four great circles drawn from a point on a sphere in relation to any great circle intersecting them all, viz. that if $ABCD, A'B'C'D'$ be two



It follows that this proposition was known before Menelaus's time. It is most easily proved by means of 'Menelaus's Theorem', III. 1, or alternatively it may be deduced from the sphere from the corresponding proposition in plane geometry just as Menelaus's theorem is transferred by him from the plane to the sphere in III. 1. We may therefore fairly conclude that both the anharmonic property and Menelaus's theorem with reference to the sphere were already in some earlier text-book; and, as Ptolemy, who built upon Hipparchus, deduces many of the trigonometric formulae which he uses from the one theorem (III. 1), Menelaus, it seems probable enough that both theorems were known to Hipparchus. The corresponding plane theorems appear in Pappus among his lemmas to Euclid's *Porisms*; there is therefore every probability that they were already known by Euclid as known.

(δ) *Propositions analogous to Eucl. VI. 3.*

Two theorems following, III. 6, 8, have their analogues in Eucl. VI. 3. In III. 6 the vertical angle A of a spherical triangle is bisected by an arc of a great circle meeting the sides AB , AC in D , and it is proved that $\sin BD / \sin DC = \sin BA / \sin AC$. In III. 8 we have the vertical angle bisected both internally and externally by arcs of great circles meeting BC in E , and the proposition proves the harmonic property

$$\frac{\sin BE}{\sin EC} = \frac{\sin BD}{\sin DC}.$$

III. 7 is to the effect that, if arcs of great circles be drawn through B to meet the opposite side AC of a spherical triangle in D , E so that $\angle ABD = \angle EBC$, then

$$\frac{\sin EA \cdot \sin AD}{\sin DC \cdot \sin CE} = \frac{\sin^2 AB}{\sin^2 BC}.$$

As this is analogous to plane propositions given by Pa

9 and III. 10 show, for a spherical triangle, that (1) the circles bisecting the three angles, (2) the great circles through the angular points meeting the opposite sides at angles meet in a point.

The remaining propositions, III. 11-15, return to the same astronomical problem as those dealt with in Euclid's *Phaenomena*, Theodosius's *Sphaerica* and Book II of Menelaus's own work. Props. 11-14 amount to theorems in spherical trigonometry such as the following.

Let arcs $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$, such that

$$90^\circ \geq \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4,$$

$$90^\circ > \beta_1 > \beta_2 > \beta_3 > \beta_4,$$

also $\alpha_1 > \beta_1, \alpha_2 > \beta_2, \alpha_3 > \beta_3, \alpha_4 > \beta_4$,

If $\sin \alpha_1 : \sin \alpha_2 : \sin \alpha_3 : \sin \alpha_4 = \sin \beta_1 : \sin \beta_2 : \sin \beta_3 : \sin \beta_4$,

$$\frac{\alpha_1 - \alpha_2}{\alpha_3 - \alpha_4} > \frac{\beta_1 - \beta_2}{\beta_3 - \beta_4}.$$

If $\frac{\sin(\alpha_1 + \beta_1)}{\sin(\alpha_1 - \beta_1)} = \frac{\sin(\alpha_2 + \beta_2)}{\sin(\alpha_2 - \beta_2)} = \frac{\sin(\alpha_3 + \beta_3)}{\sin(\alpha_3 - \beta_3)} = \frac{\sin(\alpha_4 + \beta_4)}{\sin(\alpha_4 - \beta_4)},$

$$\frac{\alpha_1 - \alpha_2}{\alpha_3 - \alpha_4} < \frac{\beta_1 - \beta_2}{\beta_3 - \beta_4}.$$

If $\frac{\sin(\alpha_1 - \alpha_2)}{\sin(\alpha_3 - \alpha_4)} < \frac{\sin(\beta_1 - \beta_2)}{\sin(\beta_3 - \beta_4)}$

$$\frac{\alpha_1 - \alpha_2}{\alpha_3 - \alpha_4} < \frac{\beta_1 - \beta_2}{\beta_3 - \beta_4}.$$

Then, given three series of three arcs such that

$$\alpha_1 > \alpha_2 > \alpha_3, \quad \beta_1 > \beta_2 > \beta_3, \quad 90^\circ > \gamma_1 > \gamma_2 > \gamma_3,$$

versal PC_3A_3 ,

$$\frac{\sin A_1A_3}{\sin BA_3} = \frac{\sin C_1C_3}{\sin BC_3} \cdot \frac{\sin PA_1}{\sin PC_1},$$

$$\frac{\sin A_1A_3}{\sin C_1C_3} = \frac{\sin PA_1}{\sin PC_1} \cdot \frac{\sin BA_3}{\sin BC_3} = \frac{\sin PA_1}{\sin PC_1} \cdot \frac{\sin PC}{\sin PC_3},$$

from above,

$$= \frac{Rr}{r_1r_3}.$$

Part 2 of the proposition proves that, if PC_2A_2 be drawn that $\sin^2 PC_2 = \sin PA_2 \cdot \sin PC$, or $r_2^2 = Rr$ (where r_2 is radius of the parallel circle through C_2), $BC_2 - BA_2$ is a minimum, while Parts 3, 4 discuss the limits to the value of ratio between the arcs A_1A_3 and C_1C_3 .

Nothing is known of the life of CLAUDIUS PTOLEMY except he was of Alexandria, made observations between the years A.D. 125 and 141 or perhaps 151, and therefore presumably wrote his great work about the middle of the reign of Emperor Trajan (A.D. 98-117) or perhaps the reign of Emperor Hadrian (A.D. 117-138). A tradition handed down by the Byzantine scholar Theodorus Meliteniota (about 1361) states that he was born, not at Alexandria, but at Ptolemais in Syria. Arabian traditions, going back probably to the 10th century, say that he lived to the age of 78, and give a number of personal details to which too much weight must not be attached.

The *Μαθηματικὴ σύνταξις* (Arab. *Almagest*).

Ptolemy's great work, the definitive achievement of Greek astronomy, bore the title *Μαθηματικῆς Συντάξεως βιβλία ιγ*, *Mathematical Collection* in thirteen Books. By the time of the commentators who distinguished the lesser treatises on astronomy forming an introduction to Ptolemy's work as *ἀστρονομούμενος (τόπος)*, the 'Little Astronomy', the great work came to be called the 'Great Collection', *μεγάλη σύνταξις*. Later still the Arabs, combining the article *Al* with

the superlative μέγιστος, made up a word Al-majisti, which became *Almagest*; and it has been known by this name ever since. The complicated character of the system expounded by Ptolemy is no doubt responsible for the fact that it speedily became the subject of elaborate commentaries.

Commentaries on the *Syntaxis*.

Pappus¹ cites a passage from his own commentary on Book I of the *Mathematica*, which evidently means Ptolemy's work. Part of Pappus's commentary on Book V, as well as his commentary on Book VI, are actually extant in the original. Theon of Alexandria, who wrote a commentary on the *Syntaxis* in eleven Books, incorporated as much as was available of Pappus's commentary on Book V with full acknowledgement, though not in Pappus's exact words. In his commentary on Book VI Theon made much more partial quotations from Pappus; indeed the greater part of the commentary on this Book is Theon's own or taken from other sources. Pappus's commentaries are called *scholia*, Theon's *ὑπομνήματα*. Passages in Pappus's commentary on Book V allude to 'the scholia preceding this one' (in the plural), and in particular to the scholium on Book IV. It is therefore all but certain that he wrote on all the Books from I to VI at least. The text of the eleven Books of Theon's commentary was published at Basel by Joachim Camerarius in 1538, but it is rare and, owing to the way in which it is printed, with insufficient punctuation marks, gaps in places, and any number of misprints, almost unusable; accordingly little attention has so far been paid to it except as regards the first two Books, which were included, in a more readable form and with a Latin translation, by Halma in his edition of Ptolemy.

Translations and editions.

The *Syntaxis* was translated into Arabic, first (we are told) by translators unnamed at the instance of Vol. 1. The 11th

The first edition to be published was the Latin translation made by Gherard of Cremona from the Arabic, which was finished in 1175 but was not published till 1515, when it was brought out, without the author's name, by Peter Liechtenstein at Venice. A translation from the Greek had been made about 1160 by an unknown writer for a certain Henricus Aristippus, Archdeacon of Catania, who, having been sent by William I, King of Sicily, on a mission to the Byzantine Emperor Manuel I. Comnenus in 1158, brought back with him a Greek manuscript of the *Syntaxis* as a present; this translation, however, exists only in manuscripts in the Vatican and at Florence. The first Latin translation from the Greek to be published was that made by Georgius 'of Trebizond' for Pope Nicolas V in 1451; this was revised and published by Lucas Gauricus at Venice in 1528. The *editio princeps* of the Greek text was brought out by Grynaeus at Basel in 1538. The next complete edition was that of Halma published 1813-16, which is now rare. All the more welcome, therefore, is the definitive Greek text of the astronomical works of Ptolemy edited by Heiberg (1899-1907), to which is now added, so far as the *Syntaxis* is concerned, a most valuable supplement in the German translation (with notes) by Manitius (Teubner, 1912-13).

Summary of Contents.

The *Syntaxis* is most valuable for the reason that it contains very full particulars of observations and investigations by Hipparchus, as well as of the earlier observations recorded by him, e.g. that of a lunar eclipse in 721 B.C. Ptolemy based himself very largely upon Hipparchus, e.g. in the preparation of a Table of Chords (equivalent to sines), the theory of eccentrics and epicycles, &c.; and it is questionable whether he himself contributed anything of great value except a definite theory of the motion of the five planets, for which Hipparchus had only collected material in the shape of observations made by his predecessors and himself. A very short indication of the subjects of the different Books is all that can

be given here. Book I: Indispensable preliminaries to the study of the Ptolemaic system, general explanations of the different motions of the heavenly bodies in relation to the earth as centre, propositions required for the preparation of Tables of Chords, the Table itself, some propositions in spherical geometry leading to trigonometrical calculations of the relations of arcs of the equator, ecliptic, horizon and meridian, a 'Table of Obliquity', for calculating declinations for each degree-point on the ecliptic, and finally a method of finding the right ascensions for arcs of the ecliptic equal to one-third of a sign or 10° . Book II: The same subject continued, i.e. problems on the sphere, with special reference to the differences between various latitudes, the length of the longest day at any degree of latitude, and the like. Book III: On the length of the year and the motion of the sun on the eccentric and epicycle hypotheses. Book IV: The length of the months and the theory of the moon. Book V: The construction of the astrolabe, and the theory of the moon continued, the diameters of the sun, the moon and the earth's shadow, the distance of the sun and the dimensions of the sun, moon and earth. Book VI: Conjunctions and oppositions of sun and moon, solar and lunar eclipses and their periods. Books VII and VIII are about the fixed stars and the precession of the equinoxes, and Books IX–XIII are devoted to the movements of the planets.

Trigonometry in Ptolemy.

What interests the historian of mathematics is the trigonometry in Ptolemy. It is evident that no part of the trigonometry, or of the matter preliminary to it, in Ptolemy was new. What he did was to abstract from earlier treatises, and to condense into the smallest possible space, the minimum of propositions necessary to establish the methods and formulae used. Thus at the beginning of the preliminaries to the Table of Chords in Book I he says:

our method of obtaining them based on geometrical considerations.'¹

He explains that he will use the division (1) of the circle into 360 equal parts or degrees and (2) of the diameter into 120 equal parts, and will express fractions of these parts on the sexagesimal system. Then come the geometrical propositions, as follows.

(a) *Lemma for finding $\sin 18^\circ$ and $\sin 36^\circ$.*

To find the side of a pentagon and decagon inscribed in a circle or, in other words, the chords subtending arcs of 72° and 36° respectively.

Let AB be the diameter of a circle, O the centre, OC the radius perpendicular to AB .

Bisect OB at D , join DC , and measure DE along DA equal to DC . Join EC .

Then shall OE be the side of the inscribed regular decagon, and EC the side of the inscribed regular pentagon.

For, since OB is bisected at D ,

$$\begin{aligned} BE \cdot EO + OD^2 &= DE^2 \\ &= DC^2 = DO^2 + OC^2. \end{aligned}$$

Therefore $BE \cdot EO = OC^2 = OB^2$,

and BE is divided in extreme and mean ratio.

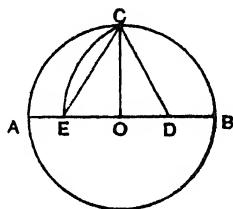
But (Eucl. XIII. 9) the sides of the regular hexagon and the regular decagon inscribed in a circle when placed in a straight line with one another form a straight line divided in extreme and mean ratio at the point of division.

Therefore, BO being the side of the hexagon, EO is the side of the decagon.

Also (by Eucl. XIII. 10)

$$\begin{aligned} (\text{side of pentagon})^2 &= (\text{side of hexagon})^2 + (\text{side of decagon})^2 \\ &= CO^2 + OE^2 = EC^2; \end{aligned}$$

therefore EC is the side of the regular pentagon inscribed in the circle.



¹ Ptolemy, *Syntaxis*, i. 10, pp. 31-2.

The construction in fact easily leads to the results

$$EO = \frac{1}{2}\alpha(\sqrt{5}-1), \quad EC = \frac{1}{2}\alpha\sqrt{(10-2\sqrt{5})},$$

where α is the radius of the circle.

Ptolemy does not however use these radicals, but calculates the lengths in terms of 'parts' of the diameter thus.

$DO = 30$, and $DO^2 = 900$; $OC = 60$ and $OC^2 = 3600$
therefore $DE^2 = DC^2 = 4500$, and $DE = 67^p 4' 55''$ nearly
therefore side of decagon or (crd. 36°) $= DE - DO = 37^p 4' 55''$

Again $OE^2 = (37^p 4' 55'')^2 = 1375.4' 15''$, and $OC^2 = 3600$
therefore $CE^2 = 4975.4' 15''$, and $CE = 70^p 32' 3''$ nearly
i.e. side of pentagon or (crd. 72°) $= 70^p 32' 3''$.

The method of extracting the square root is explained by Theon in connexion with the first of these cases, $\sqrt{4500}$ above, vol. i, pp. 61-3).

The chords which are the sides of other regular inscribed figures, the hexagon, the square and the equilateral triangle are next given, namely,

$$\text{crd. } 60^\circ = 60^p,$$

$$\text{crd. } 90^\circ = \sqrt{(2 \cdot 60^2)} = \sqrt{(7200)} = 84^p 51' 10'',$$

$$\text{crd. } 120^\circ = \sqrt{(3 \cdot 60^2)} = \sqrt{(10800)} = 103^p 55' 23''.$$

$$(\beta) \text{ Equivalent of } \sin^2 \theta + \cos^2 \theta = 1.$$

It is next observed that, if x be any arc,

$$(\text{crd. } x)^2 + \{\text{crd. } (180^\circ - x)\}^2 = (\text{diam.})^2 = 120^2,$$

a formula which is of course equivalent to $\sin^2 \theta + \cos^2 \theta = 1$.

We can therefore, from crd. 72° , derive crd. 108° , crd. 36° , crd. 144° , and so on.

(\gamma) 'Ptolemy's theorem', giving the equivalent of

proposition giving the required formula depends upon a lemma, which is the famous 'Ptolemy's theorem'.

Let a quadrilateral $ABCD$ be inscribed in a circle, the diagonals being AC , BD , to prove that

$$AC \cdot BD = AB \cdot DC + AD \cdot BC.$$

The proof is well known. Draw BE so that the angle ABE is equal to the angle DBC , and let BE meet AC in E .

Then the triangles ABE , DBC are equiangular, and therefore

$$AB : AE = BD : DC,$$

$$AB : DC = AE : BD. \quad (1)$$

Again, to each of the equal angles ABE , DBC add the angle EBD ;

the angle ABD is equal to the angle EBC , and the triangles ABD , EBC are equiangular;

therefore $BC : CE = BD : DA$,

$$AD \cdot BC = CE \cdot BD. \quad (2)$$

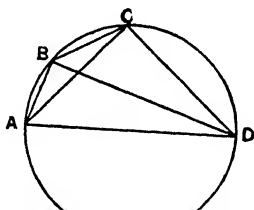
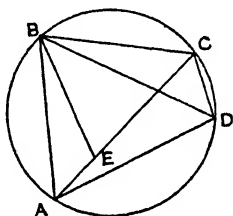
Adding (1) and (2), we obtain

$$AB \cdot DC + AD \cdot BC = AC \cdot BD.$$

Now let AB , AC be two arcs terminating at A , the extremity of the diameter AD of a circle, and let $\angle BAC (= \alpha)$ be greater than $\angle BAD (= \beta)$. Suppose that $(\text{crd. } AC)$ and $(\text{crd. } AB)$ are given: it is required to find $(\text{crd. } BC)$.

Join BD , CD .

Then, by the above theorem,



$$\{\text{crd. } (\alpha - \beta)\} \cdot (\text{crd. } 180^\circ) = (\text{crd. } \alpha) \cdot \{\text{crd. } (180^\circ - \beta)\} - (\text{crd. } \beta) \cdot \{\text{crd. } (180^\circ - \alpha)\}$$

which is, of course, equivalent to

$$\sin(\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi, \text{ where } \alpha = 2\theta, \beta = 2\phi$$

By means of this formula Ptolemy obtained

$$\text{crd. } 12^\circ = \text{crd. } (72^\circ - 60^\circ) = 12' 32'' 36'''.$$

$$(\delta) \text{ Equivalent of } \sin^2 \frac{1}{2} \theta = \frac{1}{2} (1 - \cos \theta).$$

But, in order to get the chords of smaller angles we want a formula for finding the chord of half an arc when the chord of the arc is given. This is the subject of Ptolemy's next proposition.

Let BC be an arc of a circle with diameter AC , and arc BC be bisected at D . Given $(\text{crd. } BC)$, it is required to find $(\text{crd. } DC)$.

Draw DF perpendicular to AC , and join AB, AD, BD, DC . Measure AE along AC equal to AB , and join DE .

Then shall FC be equal to EF , or FC shall be half the difference between AC and AB .

For the triangles ABD, AED are equal in all respects, since two sides of the one are equal to two sides of the other and the included angles BAD, EAD , standing on equal arcs, are equal.

$$\text{Therefore} \quad ED = BD = DC,$$

and the right-angled triangles DEF, DCF are equal in all respects, whence $EF = FC$, or $CF = \frac{1}{2}(AC - AB)$.

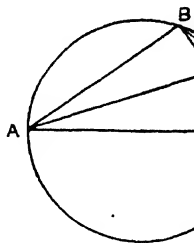
$$\text{Now} \quad AC \cdot CF = CD^2,$$

$$\text{whence } (\text{crd. } CD)^2 = \frac{1}{2} AC (AC - AB)$$

$$= \frac{1}{2} (\text{crd. } 180^\circ) \cdot \{(\text{crd. } 180^\circ) - (\text{crd. } 180^\circ - \text{crd. } BC)\}$$

This is, of course, equivalent to the formula

$$\sin^2 \frac{1}{2} \theta = \frac{1}{2} (1 - \cos \theta).$$



(*crd.* 6°), (*crd.* 3°) and finally (*crd.* $1\frac{1}{2}^\circ$) = $1^\circ 34' 15''$ and (*crd.* $\frac{3}{4}^\circ$) = $0^\circ 47' 8''$. But we want a table going by half-degrees, and hence two more things are necessary; we have to get a value for (*crd.* 1°) lying between (*crd.* $1\frac{1}{2}^\circ$) and (*crd.* $\frac{3}{4}^\circ$), and we have to obtain an *addition* formula enabling us when (*crd.* α) is given to find {*crd.* $(\alpha + \frac{1}{2}^\circ)$ }, and so on.

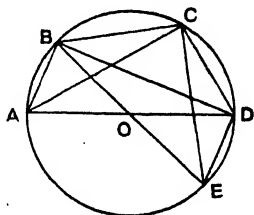
(ϵ) *Equivalent of* $\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$.

To find the addition formula. Suppose AD is the diameter of a circle, and AB, BC two arcs. Given (*crd.* AB) and (*crd.* BC), to find (*crd.* AC). Draw the diameter BOE , and join CE, CD, DE, BD .

Now, (*crd.* AB) being known, (*crd.* BD) is known, and therefore also (*crd.* DE), which is equal to (*crd.* AB); and, (*crd.* BC) being known, (*crd.* CE) is known.

And, by Ptolemy's theorem,

$$BD \cdot CE = BC \cdot DE + BE \cdot CD.$$



The diameter BE and all the chords in this equation except CD being given, we can find CD or *crd.* $(180^\circ - AC)$. We have in fact

$$\begin{aligned} &(\text{crd. } 180^\circ) \cdot \{\text{crd. } (180^\circ - AC)\} \\ &= \{\text{crd. } (180^\circ - AB)\} \cdot \{\text{crd. } (180^\circ - BC)\} - (\text{crd. } AB) \cdot (\text{crd. } BC); \end{aligned}$$

thus *crd.* $(180^\circ - AC)$ and therefore (*crd.* AC) is known.

If $AB = 2\theta$, $BC = 2\phi$, the result is equivalent to

$$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi.$$

(ξ) *Method of interpolation based on formula*

$$\sin \alpha / \sin \beta < \alpha / \beta \text{ (where } \frac{1}{2}\pi > \alpha > \beta).$$

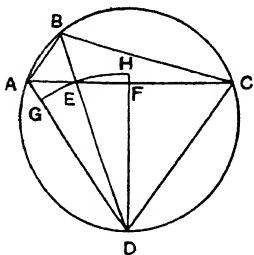
Lastly we have to find (*crd.* 1°), having given (*crd.* $1\frac{1}{2}^\circ$) and (*crd.* $\frac{3}{4}^\circ$).

Ptolemy uses an ingenious method of *interpolation* based on a proposition already assumed as known by Aristarchus.

If AB, BC be unequal chords in a circle, BC being the

ratio of the arc CB to the arc BA .

Let BD bisect the angle ABC , meeting AC in the circumference in D . Then AD, DC are then equal, and the chords AD, DC . Also C (since $CB:BA = CE:EA$).



Draw DF perpendicular then $AD > DE > DF$, so the circle with centre D and radius DF will meet DA in G and DF in H .

Now $FE:EA = \triangle FED:\triangle AED$

$< (\text{sector } HED):(\text{sector } GED)$

$< \angle FDE:\angle EDA.$

Componendo, $FA:AE < \angle FDA:\angle ADE.$

Doubling the antecedents, we have

$CA:AE < \angle CDA:\angle ADE,$

and, *separando*, $CE:EA < \angle CDE:\angle EDA;$

therefore (since $CB:BA = CE:EA$)

$CB:BA < \angle CDB:\angle BDA$

$< (\text{arc } CB):(\text{arc } BA),$

i.e. $(\text{crd. } CB):(\text{crd. } BA) < (\text{arc } CB):(\text{arc } BA).$

[This is of course equivalent to $\sin \alpha:\sin \beta < \alpha:\beta$, $\frac{1}{2}\pi > \alpha > \beta$.]

It follows (1) that $(\text{crd. } 1^\circ):(\text{crd. } \frac{3}{4}^\circ) < 1:\frac{3}{4},$

and (2) that $(\text{crd. } 1\frac{1}{2}^\circ):(\text{crd. } 1^\circ) < 1\frac{1}{2}:1.$

That is, $\frac{4}{3} \cdot (\text{crd. } \frac{3}{4}^\circ) > (\text{crd. } 1^\circ) > \frac{2}{3} \cdot (\text{crd. } 1\frac{1}{2}^\circ).$

But $(\text{crd. } \frac{3}{4}^\circ) = 0^\circ 47' 8''$, so that $\frac{4}{3}(\text{crd. } \frac{3}{4}^\circ) = 1^\circ$ nearly (actually $1^\circ 2' 50\frac{2}{3}''$);

and $(\text{crd. } 1\frac{1}{2}^\circ) = 1^\circ 34' 15''$, so that $\frac{2}{3}(\text{crd. } 1\frac{1}{2}^\circ) = 1^\circ$ nearly.

Since, then, $(\text{crd. } 1^\circ)$ is both less and greater than $\frac{2}{3}(\text{crd. } 1\frac{1}{2}^\circ)$ which only differs inappreciably from $1^\circ 2' 50''$, we may take that $(\text{crd. } 1^\circ) = 1^\circ 2' 50''$ as nearly as possible.

From this Ptolemy deduces that $(\text{crd. } \frac{1}{2}^\circ)$ is very nearly $25''$, and by the aid of the above propositions he is in position to complete his Table of Chords for arcs subtending angles increasing from $\frac{1}{2}^\circ$ to 180° by steps of $\frac{1}{2}^\circ$; in other words, a Table of Sines for angles from $\frac{1}{4}^\circ$ to 90° by steps of $\frac{1}{8}^\circ$.

(θ) *Further use of proportional increase.*

Ptolemy carries further the principle of proportional increase as a method of finding approximately the chords of arcs containing an odd number of minutes between $0'$ and $30'$. Opposite each chord in the Table he enters in a third column the excess of that chord over the one before, i.e. the chord of the arc containing $30'$ less than the chord in question. For example $(\text{crd. } 2\frac{1}{2}^\circ)$ is stated in the second column of the Table as $2^p 37' 4''$. The excess of $(\text{crd. } 2\frac{1}{2}^\circ)$ over $(\text{crd. } 2^\circ)$ in the Table is $0^p 31' 24''$; $\frac{1}{30}$ th of this is $0^p 1' 2'' 48'''$, which is therefore the amount entered in the third column opposite $(\text{crd. } 2\frac{1}{2}^\circ)$. Accordingly, if we want $(\text{crd. } 2^\circ 25')$, we take $(\text{crd. } 2^\circ)$ or $2^p 5' 40''$ and add 25 times $0^p 1' 2'' 48'''$; or we take $(\text{crd. } 2\frac{1}{2}^\circ)$ or $2^p 37' 4''$ and subtract 5 times $0^p 1' 2'' 48'''$. Ptolemy adds that if, by using the approximation for 1° and the error gradually accumulate an error, we can check the calculation by comparing the chord with that of other related arcs, the double, or the supplement (the difference between the chord and the semicircle).

Some particular results obtained from the Table may be mentioned. Since $(\text{crd. } 1^\circ) = 1^p 2' 50''$, the whole circumference is 360 ($1^p 2' 50''$), nearly, and, the length of the diameter being 120^p , the value of π is $3 (1 + \frac{2}{80} + \frac{50}{3800}) = 3 + \frac{3}{80} + \frac{30}{3800}$, which is the value used later by Ptolemy and is equivalent to $3.14166\dots$ Again, $\sqrt{3} = 2 \sin 60^\circ$ and, $2 (\text{crd. } 120^\circ)$ being equal to $2 (103^p 55' 23'')$, we have $\sqrt{3} = \frac{2}{120} (103 + \frac{55}{60} + \frac{23}{3600})$

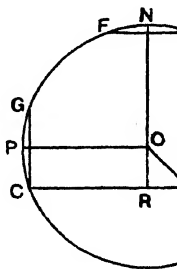
$$= 1 + \frac{43}{60} + \frac{55}{60^2} + \frac{23}{60^3} = 1.7320509,$$

which is correct to 6 places of decimals. Speaking generally,

places.

(1) *Plane trigonometry in effect used.*

There are other cases in Ptolemy in which plane trigonometry is in effect used, e.g. in the determination of the eccentricity of the sun's orbit.¹ Suppose that A is the eccentric circle with centre O , and AB , CD are chords at right angles through E , the centre of the earth. To find OE . The arc BC is known ($=\alpha$, say) as also the arc CA ($=\beta$). If BF be the chord parallel to CD , and CG the chord parallel to AB , and if N , P be the middle points of the arcs BF , GC , Ptolemy finds (1) the arc BF ($=\alpha+\beta-180^\circ$), then the chord BF , crd. $(\alpha+\beta-180^\circ)$, then the half of it, (2) the $=$ arc $(\alpha+\beta-2\beta)$ or arc $(\alpha-\beta)$, then the chord lastly half of it. He then adds the squares on the chords, i.e. he obtains



$$OE^2 = \frac{1}{4} \{ \text{crd. } (\alpha + \beta - 180) \}^2 + \frac{1}{4} \{ \text{crd. } (\alpha - \beta) \}^2$$

that is, $OE^2/r^2 = \cos^2 \frac{1}{2}(\alpha + \beta) + \sin^2 \frac{1}{2}(\alpha - \beta)$.

He proceeds to obtain the angle OEC from its sine which he expresses as a chord of double the angle in a circle on OE as diameter in relation to that diameter.

Spherical trigonometry: formulae in solution of spherical triangles.

In spherical trigonometry, as already stated, Ptolemy obtains everything that he wants by using the one fundamental proposition known as 'Menelaus's theorem' applied to the sphere (Menelaus III. 1), of which he gives the following that given by Menelaus of the first case of his proposition. Where Ptolemy has occasion for other positions of Menelaus's *Sphaerica*, e.g. III. 2 and 3,

¹ Ptolemy, *Syntaxis*, iii. 4, vol. i, pp. 234-7.

afresh by means of Menelaus's theorem.¹ The application of the theorem in other cases gives in effect the following different formulae belonging to the solution of a spherical triangle ABC right-angled at C , viz.

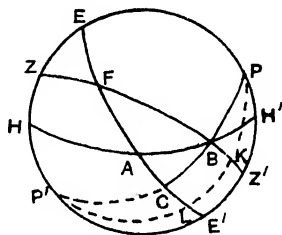
$$\sin a = \sin c \sin A,$$

$$\tan a = \sin b \tan A,$$

$$\cos c = \cos a \cos b,$$

$$\tan b = \tan c \cos A.$$

The illustration of Ptolemy's procedure will be sufficient.² Let HAH' be the horizon, $PEZH$ the meridian circle, EE' the equator, ZZ' the ecliptic, F an axial point. Let EE' , ZZ' intersect the horizon in A , B . Let P be the pole, and let the great circle PH through P , B cut the equator at C . Let it be required to find the time at which the arc FB of the ecliptic will rise; this time will be measured by the arc FA of the horizon. (Ptolemy has previously found the length of the arc BC , the declination, and FC , the right ascension, of B , § 16.)



Menelaus's theorem applied to the arcs AE' , $E'P$ cut by the arcs AH' , PC which also intersect one another in B ,

$$\frac{\text{crd. } 2PH'}{\text{crd. } 2H'E'} = \frac{\text{crd. } 2PB}{\text{crd. } 2BC} \cdot \frac{\text{crd. } 2CA}{\text{crd. } 2AE'};$$

$$\frac{\sin PH'}{\sin H'E'} = \frac{\sin PB}{\sin BC} \cdot \frac{\sin CA}{\sin AE'}.$$

Now $\sin PH' = \cos H'E'$, $\sin PB = \cos BC$, and $\sin AE' = 1$; therefore

$$\cot H'E' = \cot BC \cdot \sin CA,$$

or, in other words, in the triangle ABC right-angled at C ,

$$\cot A = \cot a \sin b,$$

$$\tan a = \sin b \tan A.$$

¹ *Syntaxis*, vol. i, p. 169 and pp. 126-7 respectively.

² *Ib.*, vol. i, pp. 121-2.

Thus AC is found, and therefore $FC - AC$ or FA .

The lengths of BC , FC are found in I. 14, 16 by the method, the four intersecting great circles used in the being in that case the equator EE' , the ecliptic ZZ' , the circle $PBCP'$ through the poles, and the great circle PL passing through the poles of both the ecliptic and the equator. In this case the two arcs PL , AE' are cut by the intersecting great circles PC , FK , and Menelaus's theorem gives (1)

$$\frac{\sin PL}{\sin KL} = \frac{\sin CP}{\sin BC} \cdot \frac{\sin BF}{\sin FK}.$$

But $\sin PL = 1$, $\sin KL = \sin BFC$, $\sin CP = 1$, $\sin FK = \sin FL$, and it follows that

$$\sin BC = \sin BF \sin BFC,$$

corresponding to the formula for a triangle right-angled at F

$$\sin a = \sin c \sin A.$$

(2) We have

$$\frac{\sin PK}{\sin KL} = \frac{\sin PB}{\sin BC} \cdot \frac{\sin CF}{\sin FL},$$

and $\sin PK = \cos KL = \cos BFC$, $\sin PB = \cos BC$, $\sin FL = \sin CF$, so that

$$\tan BC = \sin CF \tan BFC,$$

corresponding to the formula

$$\tan a = \sin b \tan A.$$

While, therefore, Ptolemy's method implicitly gives formulae for the solution of right-angled triangles, as is quoted, he does not speak of right-angled triangles at all, but only of arcs of intersecting great circles. The advantage from his point of view is that he works in sines and cosines only, avoiding tangents as such, and therefore he requires tables of only one trigonometrical ratio, namely the sine (as he has it, the chord of the double arc).

projection upon three planes mutually at right angles, the meridian, the horizon, and the 'prime vertical'. The definite problem attacked is that of showing the position of the sun at any given time of the day, and the use of the method and of the instruments described in the book by Ptolemy was connected with the construction of sundials, as we learn from Vitruvius.¹ There was another *ἀνάλημμα* besides that of Ptolemy; the author of it was Diodorus of Alexandria, a contemporary of Caesar and Cicero ('Diodorus, famed among the makers of gnomons, tell me the time!' says the Anthology²), and Pappus wrote a commentary upon it in which, as he tells us,³ he used the conchoid in order to trisect an angle, a problem evidently required in the *Analemma* in order to divide any arc of a circle into six equal parts (hours). The word *ἀνάλημμα* evidently means 'taking up' ('Aufnahme') in the sense of 'making a graphic representation' of something, in this case the representation on a plane of parts of the heavenly sphere. Only a few fragments remain of the Greek text of the *Analemma* of Ptolemy; these are contained in a palimpsest (Ambros. Gr. L. 99 sup., now 491) attributed to the seventh century but probably earlier. Besides this, we have a translation by William of Moerbeke from an Arabic version. This Latin translation was edited with a valuable commentary by the indefatigable Commandinus (Rome, 1562); but it is now available in William of Moerbeke's own words, Heiberg having edited it from Cod. Vaticanus Ottobon. lat. 1850 of the thirteenth century (written in William's own hand), and included it with the Greek fragments (so far as they exist) in parallel columns in vol. ii of Ptolemy's works (Teubner, 1907).

The figure is referred to three fixed planes (1) the meridian, (2) the horizon, (3) the prime vertical; these planes are the planes of the three circles *APZB*, *ACB*, *ZQC* respectively shown in the diagram below. Three other great circles are used, one of which, the equator with pole *P*, is fixed; the other two are movable and were called by special names; the first is the circle represented by any position of the circle of the horizon as it revolves round *COC'* as diameter (*CSM* in

¹ Vitruvius, *De architect.* ix. 4.

² *Anth. Palat.* xiv. 139.

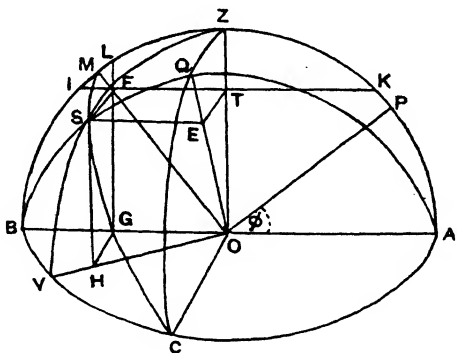
³ Pappus, iv, p. 246. 1.

and it was called *ἐκτήμερος κύκλος* ('the circle in six hours') because the highest point of it above the horizon corresponds to the lapse of six hours; the second, called the *hour-circle*, the circle represented by any position, as $BSQA$, of the circle of the horizon as it revolves round BA as axis.

The problem is, as above stated, to find the position of the sun at a given hour of the day. In order to illustrate the method, it is sufficient, with A. v. Braunmühl,¹ to consider the simplest case where the sun is on the equator, i.e. at the equinoctial points, so that the *hextemoron* circle coincides with the equator.

Let S be the position of the sun, lying on the equator, P the pole, MZA the meridian, BCA the horizon, $BSQA$ the *hour-circle*, and let the vertical great circle ZSV be drawn through S , and the vertical great circle ZQC through the zenith and C the east-point.

We are given the arc $SC = 90^\circ - t$, where t is the hour-angle, and the arc $MB = 90^\circ - \phi$, where ϕ is the elevation of the pole; and we have to find the arcs SV (the sun's altitude)



VC , the 'ascensional difference', SQ and QC . Ptolemy's method, in fact, practically determines the position of S in terms of certain spherical coordinates.

Draw the perpendiculars, SF to the plane of the meridian, SH to that of the horizon, and SE to the plane of the hour-circle.

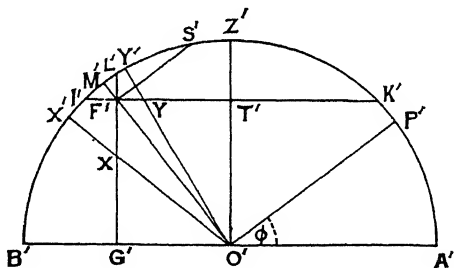
¹ Braunmühl, *Gesch. der Trigonometrie*, i, pp. 12, 13.

vertical; and draw FG perpendicular to BA , and ET to OZ . Join HG , and we have $\overline{FG} = \overline{SH}$, $\overline{GH} = \overline{FS} = \overline{ET}$.

We now represent SF in a separate figure (for clearness' sake, as Ptolemy uses only one figure), where $B'Z'A'$ corresponds to BZA , P' to P , and $O'M'$ to OM . Set off the arc $P'S'$ equal to CS ($= 90^\circ - t$), and draw $S'F'$ perpendicular to $O'M'$. Then $S'M' = SM$, and $S'F' = SF$; it is as if in the original figure we had turned the quadrant MSC round MO till it coincided with the meridian circle.

In the two figures draw IFK , $I'F'K'$ parallel to BA , $B'A'$, and LFG , $L'F'G'$ parallel to OZ , $O'Z'$.

Then (1) arc $ZI = \text{arc } ZS = \text{arc } (90^\circ - SV)$, because if we turn the quadrant ZSV about ZO till it coincides with the



meridian, S falls on I , and V on B . It follows that the required arc $SV = \text{arc } B'I'$ in the second figure.

(2) To find the arc VC , set off $G'X$ (in the second figure) along $G'F'$ equal to \overline{FS} or $\overline{F'S'}$, and draw $O'X$ through to meet the circle in X' . Then arc $Z'X' = \text{arc } VC$; for it is as if we had turned the quadrant BVC about BO till it coincided with the meridian, when (since $G'X = \overline{FS} = \overline{GH}$) H would coincide with X and V with X' . Therefore BV is also equal to $B'X'$.

(3) To find QC or ZQ , set off along $T'F'$ in the second figure $T'Y$ equal to $\overline{F'S'}$, and draw $O'Y$ through to Y' on the circle.

Then arc $B'Y' = \text{arc } QC$; for it is as if we turned the prime

both in the plane $LSHG$ at right angles to the meridian; therefore arc $SQ = \text{arc } L'Z'$.

Hence all four arcs SV , VC , QC , QS are represented in the auxiliary figure in one plane.

So far the procedure amounts to a method of *graphically* constructing the arcs required as parts of an auxiliary circle in one plane. But Ptolemy makes it clear that practical calculation followed on the basis of the figure.¹ The lines used in the construction are $SF = \sin t$ (where the radius = 1), $FT = OF \sin \phi$, $FG = OF \sin (90^\circ - \phi)$, and this was fully realized by Ptolemy. Thus he shows how to calculate the arc SZ , the zenith distance ($= d$, say) or its complement SV , the height of the sun ($= h$, say), in the following way. He says in effect: Since G is known, and $\angle F'O'G' = 90^\circ - \phi$, the ratios $O'F' : F'T'$ and $O'F' : O'T'$ are known.

[In fact $\frac{O'F'}{O'T'} = \frac{D}{\text{crd. } (180^\circ - 2\phi)}$, where D is the diameter of the sphere.]

Next, since the arc MS or $M'S'$ is known [$= t$], and therefore the arc $P'S'$ [$= 90^\circ - t$], the ratio of $O'F'$ to D is known [in fact $O'F'/D = \{\text{crd. } (180 - 2t)\} / 2D$].

It follows from these two results that

$$O'T' = \frac{\text{crd. } (180^\circ - 2t)}{2D} \cdot \text{crd. } (180^\circ - 2\phi).$$

Lastly, the arc SV ($= h$) being equal to $B'I'$, the angle h is equal to the angle $O'I'T'$ in the triangle $I'O'T'$. And in this triangle $O'I'$, the radius, is known, while $O'T'$ has been found; and we have therefore

$$\frac{O'T'}{O'I'} = \frac{\text{crd. } (2h)}{D} = \frac{\text{crd. } (180^\circ - 2t)}{D} \cdot \frac{\text{crd. } (180^\circ - 2\phi)}{D}, \text{ from above.}$$

[In other words, $\sin h = \cos t \cos \phi$; or, if $u = SC = 90^\circ - t$, $\sin h = \sin u \cos \phi$, the formula for finding $\sin h$ in the right-angled spherical triangle SVC .]

For the azimuth ω (arc $BV = \text{arc } B'Y'$) the figure gives

or $\tan VC = \tan SC \cos SCV$ in the right-angled spherical triangle SVC .

Thirdly,

$$\tan QZ = \tan Z'Y' = \frac{S'F'}{O'T'} = \frac{S'F'}{O'F'} \cdot \frac{O'F'}{O'T'} = \tan t \cdot \frac{1}{\cos \phi};$$

that is, $\frac{\tan QZ}{\tan SM} = \frac{\sin BZ}{\sin BM}$, which is Menelaus, *Sphaerica*, III. 3, applied to the right-angled spherical triangles ZBQ , MBS with the angle B common.

Zeuthen points out that later in the same treatise Ptolemy finds the arc 2α described above the horizon by a star of given declination δ' , by a procedure equivalent to the formula

$$\cos \alpha = \tan \delta' \tan \phi,$$

and this is the same formula which, as we have seen, Hipparchus must in effect have used in his *Commentary on the Phaenomena of Eudoxus and Aratus*.

Lastly, with regard to the calculations of the height h and the azimuth ω in the general case where the sun's declination is δ' , Zeuthen has shown that they may be expressed by the formulae

$$\sin h = (\cos \delta' \cos t - \sin \delta' \tan \phi) \cos \phi,$$

$$\text{and} \quad \tan \omega = \frac{\cos \delta' \sin t}{\frac{\sin \delta'}{\cos \phi} + (\cos \delta' \cos t - \sin \delta' \tan \phi) \sin \phi},$$

$$\text{or} \quad \frac{\cos \delta' \sin t}{\sin \delta' \cos \phi + \cos \delta' \cos t \sin \phi}.$$

The statement therefore of A. v. Braunmühl¹ that the Indians were the first to utilize the method of projection contained in the *Analemma* for actual trigonometrical calculations with the help of the Table of Chords or Sines requires modification in so far as the Greeks at all events showed the way to such use of the figure. Whether the practical applica-

trigonometry is the Menelaus-theorem applied to the triangle on which alone Ptolemy, as we have seen, relies in his *Syntaxis*. In any case the Table of Chords or Sines is of full use in Hipparchus's works, for it is presupposed in his method.

The *Planisphaerium*.

With the *Analemma* of Ptolemy is associated a work of somewhat similar content, the *Planisphaerium*. This again has only survived in a Latin translation of an Arabic version made by one Maslama b. Ahmad al-Maghribi of Cordova (born probably at Madrid, died 1007/8); the original is now found to be, not by Rudolph of Bruges, but by 'Hermannus Secundus', whose pupil Rudolph was the first published at Basel in 1536, and again edited, with a commentary, by Commandinus (Venice, 1558). It was later re-edited from the manuscripts by Heiberg in vol. II of the text of Ptolemy. The book is an explanation of the method of projection known as *stereographic*, by which points of the heavenly sphere are represented on the plane of the horizon by projection from one point, a pole; Ptolemy naturally takes the south pole as centre of projection, as it is the centre of the hemisphere which he is concerned to represent on the plane. Ptolemy is aware that the projections of all circles of the sphere (great circles—other than those through the centre which project into straight lines—and small circles—parallel or not parallel to the equator) are likewise circles. It is curious, however, that he does not give any proof of the fact, but is content to prove it of particular circles, such as the ecliptic, the horizon, &c. This is possible, because it is easy to show that, if a cone be cut by a plane with the pole as vertex and passing through any circle of the sphere, i.e. a circular cone, in general oblique, with the horizon as base, the section of the cone by the plane of the horizon satisfies the criterion found for the 'subcontrary section' of Apollonius at the beginning of his *Conics*, and is therefore a circle. The fact that the method of stereographic projection is so easily connected with the property of subcontrary

of oblique circular cones has led to the conjecture that Apollonius was the discoverer of the method. But Ptolemy makes no mention of Apollonius, and all that we know is that Synesius of Cyrene (a pupil of Hypatia, and born about A.D. 365-370) attributes the discovery of the method and its application to Hipparchus; it is curious that he does not mention Ptolemy's treatise on the subject, but speaks of himself alone as having perfected the theory. While Ptolemy is fully aware that circles on the sphere become circles in the projection, he says nothing about the other characteristic of this method of projection, namely that the angles on the sphere are represented by equal angles on the projection.

We must content ourselves with the shortest allusion to other works of Ptolemy. There are, in the first place, other minor astronomical works as follows:

(1) *Φάσεις ἀπλανῶν ἀστέρων* of which only Book II survives, (2) *ὑποθέσεις τῶν πλανωμένων* in two Books, the first of which is extant in Greek, the second in Arabic only, (3) the inscription in Canopus, (4) *Προχείρων κανόνων διάταξις καὶ ψηφοφορία*. All these are included in Heiberg's edition, vol. ii.

The *Optics*.

Ptolemy wrote an *Optics* in five Books, which was translated from an Arabic version into Latin in the twelfth century by a certain Admiral Eugenius Siculus¹; Book I, however, and the end of Book V are wanting. Books I, II were physical, and dealt with generalities; in Book III Ptolemy takes up the theory of mirrors, Book IV deals with concave and composite mirrors, and Book V with refraction. The theoretical portion would suggest that the author was not very proficient in geometry. Many questions are solved incorrectly owing to the assumption of a principle which is clearly false, namely that 'the image of a point on a mirror is at the point of concurrence of two lines, one of which is drawn

on the mirror where the reflection takes place'; Ptolemy gives the principle to solve various special cases of the problem (depending in general on a biquadratic equation), now known as the problem of Alhazen), 'Given a spherical surface, the position of a luminous point, and the position of a point through which the reflected ray is required, to find the point on the mirror where the reflection takes place.' Book V is the most interesting, because it may be the first attempt at a theory of refraction. It contains many details of experiments with different media, such as air and water, and gives tables of angles of refraction corresponding to different angles of incidence (i); these are based on the supposition that r and i are connected by an equation of the following form,

$$r = ai - bi^2,$$

where a , b are constants, which is worth noting as the first recorded attempt to state a law of refraction.

The discovery of Ptolemy's *Optics* in the Arab world made it clear that the work *De speculis* formerly attributed to Ptolemy is not his, and it is now practically certain that it is, at least in substance, by Heron. This is established by internal evidence, e.g. the style and certain expressions recalling others which are found in the same author's *Catoptrica* and *Dioptra*, and partly by a quotation by Heron (*On hypotheses in Optics*, chap. 14) of a proposition 'the mechanician Heron in his own *Catoptrica*', which is found in the work in question, but is not found in Ptolemy or in Euclid's. The proposition in question is that of all broken straight lines from the eye to the object, and from that again to the object, that particular broken line is shortest in which the two parts make equal angles with the surface of the mirror; the inference is that, as nature does nothing in vain, we must assume that, in reflection, the ray takes the shortest course, i.e. the angle of incidence and reflection are equal. Except for the

William of Moerbeke in 1269, was evidently made from the Greek and not from the Arabic, as is shown by Graecisms in the translation.

A mechanical work, *Περὶ ῥοπῶν*.

There are allusions in Simplicius¹ and elsewhere to a book by Ptolemy of mechanical content, *περὶ ῥοπῶν*, on balancings or turnings of the scale, in which Ptolemy maintained as against Aristotle that air or water (e.g.) in their own 'place' have no weight, and, when they are in their own 'place', either remain at rest or rotate simply, the tendency to go up or to fall down being due to the desire of things which are not in their own places to move to them. Ptolemy went so far as to maintain that a bottle full of air was not only not heavier than the same bottle empty (as Aristotle held), but actually lighter when inflated than when empty. The same work is apparently meant by the 'book on the elements' mentioned by Simplicius.² Suidas attributes to Ptolemy three Books of *Mechanica*.

Simplicius³ also mentions a single book, *περὶ διαστάσεως*, 'On dimension', i.e. dimensions, in which Ptolemy tried to show that the possible number of dimensions is limited to three.

Attempt to prove the Parallel-Postulate.

Nor should we omit to notice Ptolemy's attempt to prove the Parallel-Postulate. Ptolemy devoted a tract to this subject, and Proclus⁴ has given us the essentials of the argument used. Ptolemy gives, first, a proof of Eucl. I. 28, and then an attempted proof of I. 29, from which he deduces Postulate 5.

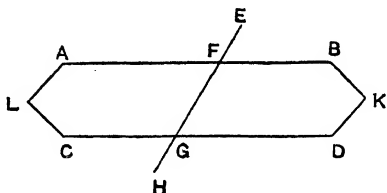
¹ Simplicius on Arist. *De caelo*, p. 710. 14, Heib. (Ptolemy, ed. Heib., vol. ii, p. 263).

² *Ib.*, p. 20. 10 sq.

³ *Ib.*, p. 9. 21 sq., (Ptolemy, ed. Heib., vol. ii, p. 265).

⁴ Proclus on Eucl. I, pp. 362. 14 sq., 365. 7-367. 27 (Ptolemy, ed. Heib., vol. ii, pp. 266-70).

I. To prove I. 28, Ptolemy takes two straight lines and a transversal $EFGH$. We have to prove that, if



of the angles BFG , FGD is equal to two right angles, then the straight lines AB , CD are parallel, i.e. non-secant.

Since AFG is the supplement of BFG , and FGC of FGD , it follows that the sum of the angles AFG , FGC is also equal to two right angles.

Now suppose, if possible, that FB , GD , making the angles BFG , FGD equal to two right angles, meet at a point. Then similarly FA , GC making the sum of the angles AFG , FGC equal to two right angles must also meet, say at point K .

[Ptolemy would have done better to point out that not only are the two sums equal but the angles themselves are equal in pairs, i.e. AFG to FGD and FGC to BFG , and therefore take the triangle KFG and apply it to FG on the other side so that the sides FK , GK may lie along GC , FA respectively, in which case GC , FA will meet at the point where K falls.]

Consequently the straight lines $LABK$, $LCDK$ enclose a space: which is impossible.

It follows that AB , CD cannot meet in either direction; they are therefore parallel.

II. To prove I. 29, Ptolemy takes two parallel lines AB , CD and the transversal FG , and argues thus. It is to be proved that $\angle AFG + \angle CGF =$ two right angles.

For, if the sum is not equal to two right angles, it is either (1) greater or (2) less.

two right angles, it must also make the other pair BFG , FGD together greater than two right angles.

But the latter pair of angles were proved less than two right angles: which is impossible.

Therefore the sum of the angles AFG , FGC cannot be greater than two right angles.

(2) Similarly we can show that the sum of the two angles AFG , FGC cannot be less than two right angles.

Therefore $\angle AFG + \angle CGF =$ two right angles.

[The fallacy here lies in the inference which I have marked by italics. When Ptolemy says that AF , CG are no more parallel than FB , GD , he is in effect assuming that *through any one point only one parallel can be drawn to a given straight line*, which is an equivalent for the very Postulate he is endeavouring to prove. The alternative Postulate is known as 'Playfair's axiom', but it is of ancient origin, since it is distinctly enunciated in Proclus's note on Eucl. I. 31.]

III. Post. 5 is now deduced, thus.

Suppose that the straight lines making with a transversal angles the sum of which is less than two right angles do not meet on the side on which those angles are.

Then, *a fortiori*, they will not meet on the other side on which are the angles the sum of which is greater than two right angles. [This is enforced by a supplementary proposition showing that, if the lines met on that side, Eucl. I. 16 would be contradicted.]

Hence the straight lines cannot meet in either direction: they are therefore *parallel*.

But in that case the angles made with the transversal are equal to two right angles: which contradicts the assumption.

Therefore the straight lines will meet.

XVIII

MENSURATION: HERON OF ALEXANDRIA

Controversies as to Heron's date.

THE vexed question of Heron's date has perhaps brought forth as much discussion as any doubtful point in the history of mathematics. In the early stages of the controversy was made of the supposed relation of Heron to Ctesibius. The *Belopoeica* of Heron has, in the best manuscript, a heading "*Ἡρώνος Κτησιβίου Βελοποιικά*", and from this, with an expression used by an anonymous Byzantine writer of the tenth century, *ὁ Ἀσκραῖος Κτησίβιος ὁ τοῦ Ἀλεξάνδρου*, "*Ἡρώνος καθηγητής*", 'Ctesibius of Askra, the teacher of Heron of Alexandria', it was inferred that Heron was a pupil of Ctesibius. The question then was, when did Ctesibius live? Martin took him to be a certain barber of that name who lived in the time of Ptolemy Euergetes II, that is, Ptolemy called Physcon (died 117 B.C.), and who is said to have invented an improved water-organ¹; Martin therefore placed Heron at the beginning of the first century (say 126–50) B.C. Philon of Byzantium, who repeatedly mentions Ctesibius by name, says that the first mechanicians (*τεχνῖται*) derived their great advantage of being under kings who loved the arts, and supported the arts.² This description applies much more to Ptolemy II Philadelphus (285–247) and Ptolemy I Soter (323–283) than to Ptolemy II Philadelphus (285–247) and Ptolemy I Soter (323–283). It is more probable, therefore, that Ctesibius was the mechanician Ctesibius who is mentioned by Aristotle as having made an elegant drinking-horn in the time of Ptolemy Philadelphus³; a pupil then of Ctesibius, Heron probably belong to the end of the third and the beginning of the second century B.C. But in truth we cannot say

above quoted, the title, however, in itself need not imply more than that Heron's work was a new edition of a similar work by Ctesibius, and the *Κτησιβίου* may even have been added by some well-read editor who knew both works and desired to indicate that the greater part of the contents of Heron's work was due to Ctesibius. One manuscript has "*Ἡρώνης Ἀλεξανδρέως Βελοποιικά*", which corresponds to the titles of the other works of Heron and is therefore more likely to be genuine.

The discovery of the Greek text of the *Metrica* by R. Schöne in 1896 made it possible to fix with certainty an upper limit. In that work there are a number of allusions to Archimedes, three references to the *χωρίου ἀποτομή* of Apollonius, and two to 'the (books) about straight lines (chords) in a circle' (*δέδεικται δὲ ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθειῶν*). Now, although the first beginnings of trigonometry may go back as far as Apollonius, we know of no work giving an actual Table of Chords earlier than that of Hipparchus. We get, therefore, at once the date 150 B.C. or thereabouts as the *terminus post quem*. A *terminus ante quem* is furnished by the date of the composition of Pappus's *Collection*; for Pappus alludes to, and draws upon, the works of Heron. As Pappus was writing in the reign of Diocletian (A.D. 284–305), it follows that Heron could not be much later than, say, A.D. 250. In speaking of the solutions by 'the old geometers' (*οἱ παλαιοὶ γεωμέτραι*) of the problem of finding the two mean proportionals, Pappus may seem at first sight to include Heron along with Eratosthenes, Nicomedes and Philon in that designation, and it has been argued, on this basis, that Heron lived long before Pappus. But a close examination of the passage¹ shows that this is by no means necessary. The relevant words are as follows:

'The ancient geometers were not able to solve the problem of the two straight lines [the problem of finding two mean proportionals to them] by ordinary geometrical methods, since the problem is by nature "solid" . . . but by attacking it with mechanical means they managed, in a wonderful way, to reduce the question to a practical and convenient construction, as may be seen in the *Mesolabon* of Eratosthenes and in the mechanics of Philon and Heron . . . Nicomedes also solved it by means of the cochloid curve, with which he also trisected an angle.'

Pappus goes on to say that he will give four solutions of which is his own; the first, second, and third he deduces as those of Eratosthenes, Nicomedes and Heron. But in an earlier sentence he mentions Philon along with Heron, and we know from Eutocius that Heron's solution is practically the same as Philon's. Hence we may conclude that by the first solution Pappus really meant Philon's, and that he omitted Heron's *Mechanics* because it was a convenient starting-point which to find the same solution.

Another argument has been based on the fact that the extracts from Heron's *Mechanics* given at the end of Book VIII, as we have it, are introduced by the author with a complaint that the copies of Heron's works in which he found them were in many respects corrupt, having lost their beginning and end.¹ But the extracts appear to have been added, not by Pappus, but by some later writer, and the argument accordingly falls to the ground.

The limits of date being then, say, 150 B.C. to A.D. 200, the only course is to try to define, as well as possible, the interval in time between Heron and the other mathematicians who may come, roughly, within the same limits. This method has been used by one of the most recent writers on the subject (Titius Livius), who places Heron not much later than 100 B.C., while another, relying almost entirely on a comparison between passages in Ptolemy and Heron, arrives at the very different conclusion that Heron was later than Ptolemy and belonged in the second century A.D.

In view of the difference between these results, it is convenient to summarize the evidence relied on to establish the earlier date, and to consider how far it is or is not conclusive against the later. We begin with the relation of Heron to Philon. Philon is supposed to come not more than a generation later than Ctesibius, because it would appear that the machines for throwing projectiles constructed by Ctesibius and Philon respectively were both available at one time for inspection by experts on the subject⁴; it is inferred

ry B.C. (If Ctesibius flourished before 247 B.C. the argument would apparently suggest rather the beginning than the end of the second century.) Next, Heron is supposed to have been a younger contemporary of Philon, the grounds being the following. (1) Heron mentions a 'stationary-automaton' representation by Philon of the Nauplius-story,¹ and this is identified by Tittel with a representation of the same story by a contemporary of Heron's (*οἱ καθ' ἡμᾶς* ²). But a careful analysis of the whole passage seems to me rather to suggest that the latter representation was not Philon's, and that Philon was included by Heron among the 'ancient' automaton-makers, and not among his contemporaries.³ (2) Another argument adduced to show that Philon was contemporary

Heron, *Autom.*, pp. 404. 11-408. 9.

² *Ib.*, p. 412. 13.

The relevant remarks of Heron are as follows. (1) He says that he found no arrangements of 'stationary automata' better or more effective than those described by Philon of Byzantium (p. 404. 11). For instance he mentions Philon's setting of the Nauplius-story, in which he found everything good except two things (a) the mechanism for the appearance of Athene, which was too difficult (*ἐργωδέστερον*), and (b) the absence of an incident promised by Philon in his description, namely the falling of a thunderbolt on Ajax with a sound of thunder accompanying it (pp. 404. 15-408. 9). This latter incident Heron could find nowhere in Philon, though he had consulted a great number of copies of his work. He continues (p. 408. 9-13) that we are not to suppose that he is running down Philon or charging him with not being able of carrying out what he promised. On the contrary, the omission is probably due to a slip of memory, for it is easy enough to make a thunder (he proceeds to show how to do it). But the rest of Philon's arrangements seemed to him satisfactory, and this, he says, is one he has not ignored Philon's work: 'for I think that my readers will derive most benefit if they are shown, first what has been well said by the ancients and then, separately from this, what the ancients overlooked and what in their work needed improvement' (pp. 408. 22-410. 6). (2) The chapter (pp. 410. 7-412. 2) explains generally the sort of thing the automaton-picture has to show, and Heron says he will give one example which he regards as the best. Then (3), after drawing a contrast between simpler pictures made by 'the ancients', which involved three different movements only, and the contemporary (*οἱ καθ' ἡμᾶς*) representations of interesting stories by means of more numerous and varied movements (pp. 412. 3-15), he proceeds to describe a setting of the Nauplius-story. This is the representation which Tittel identifies with Philon's. But it can be observed that the description includes that of the episode of the thunderbolt striking Ajax (c. 30, pp. 448. 1-452. 7) which Heron expressly says that Philon omitted. Further, the mechanism for the appearance of Athene described in c. 29 is clearly not Philon's 'more difficult' arrangement, but the simpler device described (pp. 404. 18-408. 5) as more simple and preferable to Philon's (cf. Heron, vol. i, ed. Schmidt, pp. 1-19).

with Heron is the fact that Philon has some critical details of construction of projectile-throwers which are in Heron, whence it is inferred that Philon had Heron specifically in view. But if Heron's *Βελοποιικά* was but the work of Ctesibius, it is equally possible that Philon be referring to Ctesibius.

A difficulty in the way of the earlier date is the relation which Heron stands to Posidonius. In Heron's *Mechanica* i. 24, there is a definition of 'centre of gravity' which is attributed by Heron to 'Posidonius a Stoic'. But this can hardly be Posidonius of Apamea, Cicero's teacher, because the next sentence in Heron, stating a distinction drawn by the Pythagoreans in connexion with this definition, seems to imply that the Posidonius referred to lived before Archimedes. Heron's *Definitions* do contain definitions of geometrical notions which are put down by Proclus to Posidonius of Apamea or Rhodes, and, in particular, definitions of 'centres of gravity' and of 'parallels'. Now Posidonius lived from 135 to 87 B.C. and the supporters of the earlier date for Heron cannot suggest that either Posidonius was not the first to give these definitions, or alternatively, if he was, and if they were included in Heron's *Definitions* by Heron himself and by some later editor, all that this obliges us to admit is that Heron cannot have lived before the first century B.C.

Again, if Heron lived at the beginning of the first century B.C., it is remarkable that he is nowhere mentioned by Vitruvius. The *De architectura* was apparently brought into Italy in 14 B.C. and in the preface to Book VII Vitruvius gives a list of authorities on *machinationes* from whom he has made extracts. The list contains twelve names and has the appearance of being scrupulously complete; but, while it includes Archytas (second), Archimedes (third), Ctesibius (fourth), and Philon of Byzantium (sixth), it does not mention Heron. Nor is it possible to establish interdependence between Heron and Vitruvius; the differences seem,

machines used by the two for the same purpose frequently differ in details; e.g. in Vitruvius's hodometer a pebble drops into a box at the end of each Roman mile,¹ while in Heron's the distance completed is marked by a pointer.² It is indeed pointed out that the water-organ of Heron is in many respects more primitive than that of Vitruvius; but, as the instruments are altogether different, this can scarcely be said to prove anything.

On the other hand, there are points of contact between certain propositions of Heron and of the Roman *agrimensores*. Columella, about A.D. 62, gave certain measurements of plane figures which agree with the formulae used by Heron, notably those for the equilateral triangle, the regular hexagon (in this case not only the formula but the actual figures agree with Heron's) and the segment of a circle which is less than a semicircle, the formula in the last case being

$$\frac{1}{2}(s+h)h + \frac{1}{14}\left(\frac{1}{2}s\right)^2,$$

where s is the chord and h the height of the segment. Here there might seem to be dependence, one way or the other; but the possibility is not excluded that the two writers may merely have drawn from a common source; for Heron, in giving the formula for the area of the segment of a circle, states that it was the formula used by 'the more accurate investigators' (*οἱ ἀκριβέστερον ἐζητηκότες*).³

We have, lastly, to consider the relation between Ptolemy and Heron. If Heron lived about 100 B.C., he was 200 years earlier than Ptolemy (A.D. 100–178). The argument used to prove that Ptolemy came some time after Heron is based on a passage of Proclus where Ptolemy is said to have remarked on the untrustworthiness of the method in vogue among the 'more ancient' writers of measuring the apparent diameter of the sun by means of water-clocks.⁴ Hipparchus, says Proclus, used his dioptra for the purpose, and Ptolemy followed him. Proclus proceeds:

Hipparchus. And first we will show how we can measure an interval of time by means of the regular efflux of a procedure which was explained by Heron the mechanic in his treatise on water-clocks.'

Theon of Alexandria has a passage to a similar effect. He first says that the most ancient mathematicians constructed a vessel which would let water flow out uniformly through a small aperture at the bottom, and then adds at the end, in the same words as Proclus uses, that Heron showed how this is managed in the first book of his work on water-clocks. Theon's account is from Pappus's *Commentary on the Syntaxis*, and this is also Proclus's source, as is shown by the fact that Proclus gives a drawing of the water-clock which appears to have been lost in Theon's transcription. Pappus, but which Pappus must have reproduced from the work of Heron. Tittel infers that Heron must have been regarded as one of the 'more ancient' writers as compared with Ptolemy. But this again does not seem to be a necessary inference. No doubt Heron's work was a convenient point of reference to refer to for a description of a water-clock, but it does not necessarily follow that Ptolemy was referring to Heron's clock rather than some earlier form of the same instrument.

An entirely different conclusion from that of Theon is reached in the article 'Ptolemaios and Heron' already mentioned.² The arguments are shortly these. (1) Ptolemy's method in his *Geography* (c. 3) that his predecessors had only been able to measure the distance between two places (as an arc of a great circle on the earth's circumference) in the case where the two places are on the same meridian. He claims that he himself invented a way of doing this even in the case where the two places are neither on the same meridian nor on the same parallel circle, provided that the heights of the sun at the two places respectively, and the angle between the meridian circle passing through both and the meridian circle through one of the places, are known. Now Heron in his *Measurement*

the distance between Rome and Alexandria.¹ Unfortunately the text is in places corrupt and deficient, so that the method cannot be reconstructed in detail. But it involved the observation of the same lunar eclipse at Rome and Alexandria respectively and the drawing of the *analemma* for Rome. That is to say, the mathematical method which Ptolemy claims to have invented is spoken of by Heron as a thing generally known to experts and not more remarkable than other technical matters dealt with in the same book. Consequently Heron must have been later than Ptolemy. (It is right to add that some hold that the chapter of the *Dioptra* in question is not germane to the subject of the treatise, and was probably not written by Heron but interpolated by some later editor; if this is so, the argument based upon it falls to the ground.) (2) The dioptra described in Heron's work is a fine and accurate instrument, very much better than anything Ptolemy had at his disposal. If Ptolemy had been aware of its existence, it is highly unlikely that he would have taken the trouble to make his separate and imperfect 'parallactic' instrument, since it could easily have been grafted on to Heron's dioptra. Not only, therefore, must Heron have been later than Ptolemy but, seeing that the technique of instrument-making had made such strides in the interval, he must have been considerably later. (3) In his work *περὶ ῥοπῶν*² Ptolemy, as we have seen, disputed the view of Aristotle that air has weight even when surrounded by air. Aristotle satisfied himself experimentally that a vessel full of air is heavier than the same vessel empty; Ptolemy, also by experiment, convinced himself that the former is actually the lighter. Ptolemy then extended his argument to water, and held that water with water round it has no weight, and that the diver, however deep he dives, does not feel the weight of the water above him. Heron³ asserts that water has no appreciable weight and has no appreciable power of compressing the air in a vessel inverted and forced down into the water. In confirmation of this he cites the case of the

diver is not oppressed though he has an unlimited weight of water on his back. He accepts, therefore, the view of Pappus as to the fact, however strange this may seem. But he is not satisfied with the explanation given: 'Some say', he goes on, 'it is because water in itself is uniformly heavy (*ἰσοβαρὲς καθ' αὐτό*)'—this seems to be equivalent to Ptolemy's statement that water in water has no weight—'but they give no explanation whatever why divers . . .' He himself attempts an explanation based on Archimedes. It is suggested, therefore, that Heron's criticism is directed specifically against Pappus and no one else. (4) It is suggested that the Dionysius to whom Heron dedicated his *Definitions* is a certain Dionysius who was *praefectus urbi* at Rome in A.D. 301. The grounds for these. (a) Heron addresses Dionysius as *Διονύσιε λαμπρότατος*, where *λαμπρότατος* obviously corresponds to the Latin *clarissimus*, a title which in the third century and under Diocletian was not yet in common use. Further, this Dionysius is identified as *curator aquarum* and *curator operum publicorum*, so that he was the sort of person who would have to do with the work of engineers, architects and craftsmen for whom Heron wrote. Lastly, he is mentioned in an inscription commemorating the improvement of water supply and dedicated 'to Tibullus, father of all waters, and to the ancient inventors of numerous constructions' (*repertoribus admirabilium fabricarum priscis viris*), an expression which is not found in any inscription, but which recalls the sort of tribute that the emperor frequently pays to his predecessors. This identification of two persons named Dionysius is an ingenious conjecture, but the evidence is not such as to make it anything more.¹

The result of the whole investigation just summarized is to place Heron in the third century A.D., and perhaps later than anything, earlier than Pappus. Heiberg accepts this conclusion,² which may therefore, I suppose, be said to hold true for the present.

¹ Dionysius was of course a very common name. Diophantus dedicated his *Arithmetica* to a person of this name (*τιμωτάτῳ μοι Διονύσιῳ*), who is praised for his ambition to learn the solutions of arithmetical problems. This Dionysius must have lived in the second half of the third century A.D., and if Heron also belonged to this time, is it not possible that Heron's Dionysius was the same person?

² Heron, vol. v, p. ix.

Heron was known as ὁ Ἀλεξανδρεὺς (e.g. by Pappus) or ὁ μηχανικός (*mechanicus*), to distinguish him from other persons of the same name; Proclus and Damianus use the latter title, while Pappus also speaks of οἱ περὶ τὸν Ἡρώνα μηχανικοί.

Character of works.

Heron was an almost encyclopaedic writer on mathematical and physical subjects. Practical utility rather than theoretical completeness was the object aimed at; his environment in Egypt no doubt accounts largely for this. His *Metrica* begins with the old legend of the traditional origin of geometry in Egypt, and in the *Dioptra* we find one of the very problems which geometry was intended to solve, namely that of re-establishing boundaries of lands when the flooding of the Nile had destroyed the land-marks: 'When the boundaries of an area have become obliterated to such an extent that only two or three marks remain, in addition to a plan of the area, to supply afresh the remaining marks.'¹ Heron makes little or no claim to originality; he often quotes authorities, but, in accordance with Greek practice, he more frequently omits to do so, evidently without any idea of misleading any one; only when he has made what is in his opinion any slight improvement on the methods of his predecessors does he trouble to mention the fact, a habit which clearly indicates that, except in these cases, he is simply giving the best traditional methods in the form which seemed to him easiest of comprehension and application. The *Metrica* seems to be richest in definite references to the discoveries of predecessors; the names mentioned are Archimedes, Dionysodorus, Eudoxus, Plato; in the *Dioptra* Eratosthenes is quoted, and in the introduction to the *Catoptrica* Plato and Aristotle are mentioned.

The practical utility of Heron's manuals being so great, it was natural that they should have great vogue, and equally natural that the most popular of them at any rate should be

books in particular gave scope for expansion by multiplication of examples, so that it is difficult to disentangle the Heron from the rest of the collections which have come to us under his name. Hultsch's considered criteria follows: 'The Heron texts which have come down to our time are authentic in so far as they bear the author's stamp and have kept the original design and form of Heron's works, but are unauthentic in so far as, being constantly in use for practical purposes, they were repeatedly re-edited and in the course of re-editing, were rewritten with a view to the particular needs of the time.'

List of Treatises.

Such of the works of Heron as have survived have come to us in very different ways. Those which have come down to us in the Greek are:

I. The *Metrica*, first discovered in 1896 in a manuscript of the eleventh (or twelfth) century at Constantinople. It was edited by R. Schöne and edited by his son, H. Schöne (*Heronis Opera*, Teubner, 1903).

II. *On the Dioptra*, edited in an Italian version by L. B. B. in 1814; the Greek text was first brought out by J. Vincent¹ in 1858, and the critical edition of it by H. Schöne is included in the Teubner vol. iii just mentioned.

III. The *Pneumatica*, in two Books, which appeared in a Latin translation by Commandinus, published at his death in 1575; the Greek text was first edited by Th. Heiberg in *Veterum mathematicorum opera Graece et Latine* (Paris, 1693), and is now available in *Heronis Opera*, Teubner, 1899), by W. Schmidt.

IV. *On the art of constructing automata* (περὶ αὐτοματικῆς), or *The automaton-theatre*, first edited in an Italian translation by B. Baldi in 1589; the Greek text was first printed in Thévenot's *Vet. math.*, and now forms part of *Heronis Opera*, vol. i, by W. Schmidt.

V. *Belopoeica* (on the construction of engines of war)

¹ *Notices et extraits des manuscrits de la Bibliothèque impériale*, pp. 157-337.

by B. Baldi (Augsburg, 1616), Thévenot (*Vet. math.*), Köchly and Rüstow (1853) and by Wescher (*Poliorcétique des Grecs*, 1867, the first critical edition).

VI. The *Cheirobalistra* ("Ἡρώων χειροβαλλίστρας κατασκευὴ καὶ συμμετρία (?)), edited by V. Prou, *Notices et extraits*, xxvi. 2 (Paris, 1877).

VII. The geometrical works, *Definitiones, Geometria, Geodaesia, Stereometrica* I and II, *Mensurae, Liber Geeponicus*, edited by Hultsch with *Variae collectiones (Heronis Alexandrini geometricorum et stereometricorum reliquiae*, 1864). This edition will now be replaced by that of Heiberg in the Teubner collection (vols. iv, v), which contains much additional matter from the Constantinople manuscript referred to, but omits the *Liber Geeponicus* (except a few extracts) and the *Geodaesia* (which contains only a few extracts from the *Geometry* of Heron).

Only fragments survive of the Greek text of the *Mechanics* in three Books, which, however, is extant in the Arabic (now edited, with German translation, in *Heronis Opera*, vol. ii, by L. Nix and W. Schmidt, Teubner, 1901).

A smaller separate mechanical treatise, the *Βαρουλκός*, is quoted by Pappus.¹ The object of it was 'to move a given weight by means of a given force', and the machine consisted of an arrangement of interacting toothed wheels with different diameters.

At the end of the *Dioptra* is a description of a *hodometer* for measuring distances traversed by a wheeled vehicle, a kind of taxameter, likewise made of a combination of toothed wheels:

A work on *Water-clocks* (περὶ ὑδρίων ὥροσκοπέων) is mentioned in the *Pneumatica* as having contained four Books, and is also alluded to by Pappus.² Fragments are preserved in Proclus (*Hypotyposis*, chap. 4) and in Pappus's commentary on Book V of Ptolemy's *Syntaxis* reproduced by Theon.

Of Heron's *Commentary on Euclid's Elements* only very meagre fragments survive in Greek (Proclus), but a large

Besthorn and Heiberg (*Codex Leidensis* 399. 1, five which had appeared up to 1910). The commentary as far as *Elem.* VIII. 27 at least.

The *Catoptrica*, as above remarked under Ptolemy, a Latin translation from the Greek, presumed to be by Moerbeke, and is included in vol. ii of *Heroni* edited, with introduction, by W. Schmidt.

Nothing is known of the *Camarica* ('on vaulting' mentioned by Eutocius (on Archimedes, *Sphere and Cylinder*), *Zygia* (balancings) associated by Pappus with the *Automata* or of a work on the use of the astrolabe mentioned in the *Fihrist*.

We are in this work concerned with the treatises of mathematical content, and therefore can leave out of account works as the *Pneumatica*, the *Automata*, and the *Ballistics*. The *Pneumatica* and *Automata* have, however, an interest to the historian of physics in so far as they employ the compressed air, water, or steam. In the *Pneumatica* the reader will find such things as siphons, 'Heron's fountain', 'penny-in-the-slot' machines, a fire-engine, a water-organ, and many arrangements employing the force of steam.

Geometry.

(a) *Commentary on Euclid's Elements.*

In giving an account of the geometry and mechanics (or geodesy) of Heron it will be well, I think, to begin with what relates to the *elements*, and first the commentary on Euclid's *Elements*, of which we possess a number of extracts in an-Nairizī and Proclus, enabling us to get a general idea of the character of the work. Generally, Heron's comments do not appear to have much that can be called important. They may be summarised as follows :

- (1) A few general notes, e.g. that Heron would not accept more than three axioms.

the chord is to be compared are drawn on different sides of the diameter instead of on the same side), III. 12 (which is Euclid's at all but Heron's own, adding the case of external to that of internal contact in III. 11), VI. 19 (where a triangle in which an additional line is drawn is taken to be the *smaller* of the two), VII. 19 (where the particular case is given of *three* numbers in continued proportion instead of proportionals).

Alternative proofs.

It appears to be Heron who first introduced the easy but destructive semi-algebraical method of proving the propositions II. 2-10 which is now so popular. On this method the propositions are proved 'without figures' as consequences of correspondences to the algebraical formula

$$a(b + c + d + \dots) = ab + ac + ad + \dots$$

Heron explains that it is not possible to prove II. 1 without using a number of lines (i.e. without actually drawing the angles), but that the following propositions up to II. 10 can be proved by merely drawing one line. He distinguishes two varieties of the method, one by *dissolutio*, the other by *compositio*, by which he seems to mean *splitting-up* of rectangles and squares and *combination* of them into others. In his proofs he sometimes combines the two varieties.

Alternative proofs are given (a) of some propositions of Book III, namely III. 25 (placed after III. 30 and starting with the *arc* instead of the chord), III. 10 (proved by means of I. 9), III. 13 (a proof preceded by a lemma to the effect that a straight line cannot meet a circle in more than two points).

The second class of alternative proof is (b) that which is intended to meet a particular objection (*ἐνστάσις*) which had been or might be raised to Euclid's constructions. Thus in certain cases Heron avoids *producing* a certain straight line, where Euclid produces it, the object being to meet the objection of one who might deny our right to assume that there is *any space available*. Of this class are his proofs of I. 11, 20 and his proof on I. 16. Similarly in I. 48 he supposes the right-angled

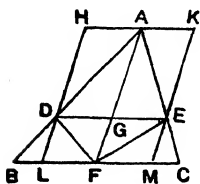
triangle which is constructed to be constructed on the side of the common side as the given triangle is.

A third class (c) is that which avoids *reductio ad absurdum* e.g. a direct proof of I. 19 (for which he requires and gives a preliminary lemma) and of I. 25.

(4) Heron supplies certain *converses* of Euclid's propositions e.g. of II. 12, 13 and VIII. 27.

(5) A few additions to, and extensions of, Euclid's propositions are also found. Some are unimportant, e.g. the construction of isosceles and scalene triangles in a note on I. 1 and construction of *two* tangents in III. 17. The most important extension is that of III. 20 to the case where the angle at the circumference is greater than a right angle, which gives an easy way of proving the theorem of III. 22. Interesting are the notes on I. 37 (on I. 24 in Proclus), where Heron proves that two triangles with two sides of the one equal to two sides of the other and with the included angles *supplementary* are equal in area, and compares the areas where the sum of the included angles (one being supposed greater than the other) is less or greater than two right angles, and on I. 47 where there is a proof (depending on preliminary lemmas) of the fact that, in the figure of Euclid's proposition (see page 100), the straight lines AL , BG , CE meet in a point. The last proof is worth giving. First come the lemmas.

(1) If in a triangle ABC a straight line DE be drawn parallel to the base BC cutting the sides AB , AC or the



sides produced in D , E , and if F be the middle point of BC , then the straight line AF (produced if necessary) will also bisect DE . (HK is drawn through A parallel to DE , and HDL , KEM through D , E parallel to AF meeting the base in L , M respectively. Then the triangles ABF , ACE are equal. So are the triangles ABD , ACE . The triangles ABF , ACE are equal. So are the triangles ABD , ACE .

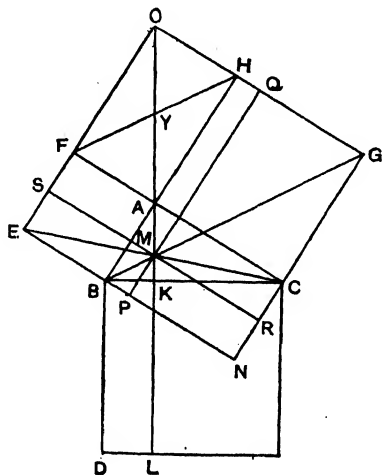
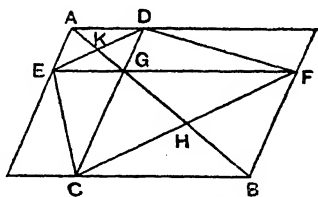
equal, the common vertex G will lie on the diagonal AB . Heron produces AG to meet CF in H , and then proves that CH is a straight line.

Since DF, CE are equal, so are the triangles DGF, ECG . Adding triangle GCF , we have the angles ECF, DCF equal, and CE, CF are parallel.

But (by I. 34, 29, 26) the triangles AKE, GKD are congruent, so that $EK = KD$; and by lemma (1) it follows that $CH = HF$. Now, in the triangles FHB, CHG , two sides (BF, FH and CH) and the included angles are equal; therefore the triangles are congruent, and the angles BHF, GHC are equal. Add to each the angle GHF , and

$BHF + \angle FHG = \angle CHG + \angle GHF = \text{two right angles}.$

To prove his substantive proposition Heron draws AKL perpendicular to BC , and joins EC meeting AK in M . Then he has only to prove that BMG is a straight line.



complete the parallelogram $FAHO$, and draw the diagonals FH meeting in Y . Through M draw PQ, SR parallel respectively to BA, AC .

Now the triangles FAH , BAC are equal in all respects;
 therefore $\angle HFA = \angle ABC$
 $= \angle CAK$ (since AK is at right angles to BC).

But, the diagonals of the rectangle FH cutting one another in Y , we have $FY = YA$ and $\angle HFA = \angle OAF$;
 therefore $\angle OAF = \angle CAK$, and OA is in a straight line with AKL .

Therefore, OM being the diagonal of SQ , $SA = AQ$, and, if we add AM to each, $FM = MH$.

Also, since EC is the diagonal of FN , $FM = MN$.

Therefore the parallelograms MH , MN are equal; and hence, by the preceding lemma, BMG is a straight line. Q.E.D.

(β) The *Definitions*.

The elaborate collection of *Definitions* is dedicated to one Dionysius in a preface to the following effect:

'In setting out for you a sketch, in the shortest possible form, of the technical terms premised in the elements of geometry; I shall take as my point of departure, and shall base my whole arrangement upon, the teaching of Euclid, the author of the elements of theoretical geometry; for by this means I think that I shall give you a good general understanding not only of Euclid's doctrine but of many other works in the domain of geometry. I shall begin then with the *point*.'

He then proceeds to the definitions of the point, the line, the different sorts of lines, straight, circular, 'curved' and 'spiral-shaped' (the Archimedean spiral and the cylindrical helix), Defs. 1-7; surfaces, plane and not plane, solid body, Defs. 8-11; angles and their different kinds, plane, solid, rectilinear and not rectilinear, right, acute and obtuse angles, Defs. 12-22; figure, boundaries of figure, varieties of figure, plane, solid, composite (of homogeneous or non-homogeneous parts) and incomposite, Defs. 23-6. The incomposite plane

composite or homogeneous parts) and *αἰε* (*hēkēty*), bounded by four circular arcs, two concave and two convex, Defs. 27–38. Rectilineal figures follow, the various kinds of triangles and of quadrilaterals, the gnomon in a parallelogram, and the gnomon in the more general sense of the figure which added to a given figure makes the whole into a similar figure, polygons, the parts of figures (side, diagonal, height of a triangle), perpendicular, parallels, the three figures which will fill up the space round a point, Defs. 39–73. Solid figures are next classified according to the surfaces bounding them, and lines on surfaces are divided into (1) simple and circular, (2) mixed, like the conic and spiric curves, Defs. 74, 75. The sphere is then defined, with its parts, and stated to be the figure which, of all figures having the same surface, is the greatest in content, Defs. 76–82. Next the cone, its different species and its parts are taken up, with the distinction between the three conics, the section of the acute-angled cone ('by some also called *ellipse*') and the sections of the right-angled and obtuse-angled cones (also called *parabola* and *hyperbola*), Defs. 83–94; the cylinder, a section in general, the *spire* or *tore* in its three varieties, open, continuous (or just closed) and 'crossing-itself', which respectively have sections possessing special properties, 'square rings' which are cut out of cylinders (i.e. presumably rings the cross-section of which through the centre is two squares), and various other figures cut out of spheres or mixed surfaces, Defs. 95–7; rectilineal solid figures, pyramids, the five regular solids, the semi-regular solids of Archimedes two of which (each with fourteen faces) were known to Plato, Defs. 98–104; prisms of different kinds, parallelepipeds, with the special varieties, the cube, the *beam*, *δοκός* (length longer than breadth and depth, which may be equal), the *brick*, *πλινθίς* (length less than breadth and depth), the *σφηνίσκος* or *βωμίσκος* with length, breadth and depth unequal, Defs. 105–14.

Lastly come definitions of relations, equality of lines, surfaces, and solids respectively, similarity of figures, 'reciprocal figures', Defs. 115–18; indefinite increase in magnitude, parts (which must be homogeneous with the wholes, so that e.g. the horn-like angle is not a part or submultiple of a right

or any angle), multiples, Defs. 119–21; proportion in tudes, what magnitudes can have a ratio to one another, magnitudes in the same ratio or magnitudes in proportion, definition of greater ratio, Defs. 122–5; transformation of ratios (*componendo, separando, convertendo, alternando, vertendo* and *ex aequali*), Defs. 126–7; commensurable and incommensurable magnitudes and straight lines, Defs. 128–129. There follow two tables of measures, Defs. 130–2.

The *Definitions* are very valuable from the point of view of the historian of mathematics, for they give the different native definitions of the fundamental conceptions; they also find the Archimedean ‘definition’ of a straight line, and definitions which we know from Proclus to be due to Posidonius, others from Posidonius, and so on. No doubt the collection may have been recast by some editor or compiler after Heron’s time, but it seems, at least in substance, to go back to Heron or earlier still. So far as it contains definitions of Posidonius, it cannot have been compiled later than the first century B.C.; but its content seems to belong to the main to the period before the Christian era. Heiberg adds to his edition of the *Definitions* extracts from Heron’s *Geometry*, postulates and axioms from Euclid, extracts from Geminus on the classification of mathematics, the principles of geometry, &c., extracts from Proclus or some early commentator on the scholia on Euclid, and extracts from Anatolius of Theon of Smyrna, which followed the actual definitions in the manuscripts. These various additions were apparently collected by some Byzantine editor, perhaps of the eleventh century.

Mensuration.

The *Metrica, Geometrica, Stereometrica, Geodaestica, Mensurae*.

We now come to the mensuration of Heron. Under the different works under this head the *Metrica* is the most important from our point of view because it seems, more

may then in particular cases be taken to be feet, cubits, any other unit of measurement. Up to 1896, when a manuscript of it was discovered by R. Schöne at Constantinople, it was only known by an allusion to it in Eutocius (Archimedes's *Measurement of a Circle*), who states that the way to obtain an approximation to the square root of a square number is shown by Heron in his *Metrica*, as well as by Pappus, Theon, and others who had commented on the *Syntaxis* of Ptolemy.¹ Tannery² had already in 1894 discovered a fragment of Heron's *Metrica* giving the particular method in a Paris manuscript of the thirteenth century containing Prolegomena to the *Syntaxis* compiled presumably from commentaries of Pappus and Theon. Another interesting difference between the *Metrica* and the other works is that in the former the Greek way of writing fractions (which is our mode) largely preponderates, the Egyptian form (which expresses a fraction as the sum of diminishing submultiples) being used comparatively rarely, whereas the reverse is the case in the other works.

In view of the greater authority of the *Metrica*, we shall take it as the basis of our account of the mensuration, while regarding the other works in view. It is desirable at the present time to compare broadly the contents of the various collections. Book I of the *Metrica* contains the mensuration of circles, rectangles and triangles (chaps. 1-9), parallel-trapezia, rhombi, rhomboids and quadrilaterals with one angle right (10-16), regular polygons from the equilateral triangle to the regular dodecagon (17-25), a ring between two concentric circles (26), segments of circles (27-33), an ellipse (34), a parabolic segment (35), the surfaces of a cylinder (36), an isosceles cone (37), a sphere (38) and a segment of a sphere (39). Book II gives the mensuration of certain solids, the solid content of a cone (chap. 1), a cylinder (2), rectilinear solid content, a parallelepiped, a prism, a pyramid and a frustum (3-8), a frustum of a cone (9, 10), a sphere and a segment of a sphere (11, 12), a *spire* or *tore* (13) the section of a cylinder measured in Archimedes's *Method* (14), and the solid

¹ Archimedes, vol. iii, p. 232. 13-17.

² Tannery, *Mémoires scientifiques*, ii, 1912, pp. 447-54.

formed by the intersection of two cylinders with axes at right angles inscribed in a cube, also measured in the *Metrica* among the five regular solids (16-19). Book III deals with the division of figures into parts having given ratios to one another, first plane figures (1-19), then solids, a pyramid, a conical frustum, a sphere (20-3).

The *Geometria* or *Geometrumena* is a collection based on the work of Heron, but not his work in its present form. The addition of a theorem due to Patricius¹ and a reference to him in the *Stereometrica* (I. 22) suggest that Patricius edited both, but the date of Patricius is uncertain. Tannery identified him with a mathematical professor of the tenth century, Nicephorus Patricius; if this is correct, he would be contemporary with the Byzantine writer (erroneously called Pappus) who is known to have edited genuine works of Heron. It is probable that indeed Patricius and the anonymous Byzantine might be the same person. The mensuration in the *Geometria* refers almost entirely to the same figures as those measured in Book I of the *Metrica*, the difference being that (1) the rules are not explained but only applied to examples, (2) a large number of numerical calculations are given for each figure, (3) the Egyptian method of writing fractions as the sum of submultiples is followed, (4) lengths and areas are given in terms of practical measures, and the calculations are lengthened by a considerable amount of conversion from one measure into another. The first chapters (1-4) are of the nature of a general introduction, including certain definitions and ending with a list of measures. Chaps. 5-99, Hultsch (= 5-20, 14, Heib.), are for the most part corresponding in content to the *Metrica*, but seem to have been based on a different collection, while chaps. 100-3 and 105 (= 21, 1-25, 22, 3-24, Heib.) are modelled on the *Metrica*, and 101 is headed 'A demonstration (really 'measurement') of a circle in another book of Heron'. Heiberg transfers to the *Geometrica* a considerable amount of the content of the so-called *Liber Geometricus*, a badly

Heiberg puts side by side with corresponding sections of the *Geometrica* in parallel columns; others he inserts in suitable places; sections 78, 79 contain two important problems in indeterminate analysis (= *Geom.* 24, 1-2, Heib.). Heiberg adds, from the Constantinople manuscript containing the *Metrica*, eleven more sections (chap. 24, 3-13) containing indeterminate problems, and other sections (chap. 24, 14-30 and 37-51) giving the mensuration, mainly, of figures inscribed in or circumscribed to others, e. g. squares or circles in triangles, circles in squares, circles about triangles, and lastly of circles and segments of circles.

The *Stereometrica* I has at the beginning the title *Εἰσαγωγή τῶν στερεομετρούμενων Ἡρώων* but, like the *Geometrica*, seems to have been edited by Patricius. Chaps. 1-40 give the mensuration of the geometrical solid figures, the sphere, the cone, the frustum of a cone, the obelisk with circular base, the cylinder, the 'pillar', the cube, the *σφηνίσκος* (also called *δνυξ*), the *μείουρον προσκαρφευμένον*, pyramids, and frusta. Some portions of this section of the book go back to Heron; thus in the measurement of the sphere chap. 1 = *Metrica* II. 11, and both here and elsewhere the ordinary form of fractions appears. Chaps. 41-54 measure the contents of certain buildings or other constructions, e. g. a theatre, an amphitheatre, a swimming-bath, a well, a ship, a wine-butt, and the like.

The second collection, *Stereometrica* II, appears to be of Byzantine origin and contains similar matter to *Stereometrica* I, parts of which are here repeated. Chap. 31 (27, Heib.) gives the problem of Thales, to find the height of a pillar or a tree by the measurement of shadows; the last sections measure various pyramids, a prism, a *βωμίσκος* (little altar).

The *Geodaesia* is not an independent work, but only contains extracts from the *Geometry*; thus chaps. 1-16 = *Geom.* 5-31, Hultsch (= 5, 2-12, 32, Heib.); chaps. 17-19 give the methods of finding, in any scalene triangle the sides of which

not in its present form be due to Heron, although possibly it have points of contact with the genuine works. Sections 1-27 measure all sorts of objects, e.g. stones of different shapes, a pillar, a tower, a theatre, a ship, a vault, a hippodrome, etc. Sects. 28-35 measure geometrical figures, a circle and sectors of a circle (cf. *Metrica* I), and sects. 36-48 on spheres, cylinders, cones, and frusta are closely connected with *Stereom.* I and *Metrica* II; sects. 49-59, giving the measurement of receptacles and plane figures of various shapes, seem to have a different origin. We can now take up

Contents of the *Metrica*.

Book I. Measurement of Areas.

The preface records the tradition that the first systematic method of measuring land arose out of the practical necessity of measuring and distributing land (whence the name 'geometry'), and that the extension to three dimensions became necessary in order to measure solid bodies. Heron then mentions Eudoxus and Archimedes as pioneers in the discovery of difficult problems, Eudoxus having been the first to prove that the sphere is three times the cone on the same base and of equal height, and that circles are to one another as the squares on their diameters, while Archimedes first proved that the surface of a sphere is equal to four times the area of a great circle, and the volume two-thirds of the cylinder circumscribed about it.

(α) *Area of scalene triangle.*

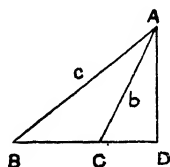
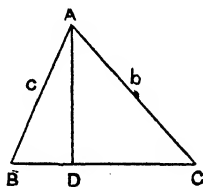
After the easy cases of the rectangle, the right-angled triangle and the isosceles triangle, Heron gives two methods of finding the area of a scalene triangle (acute-angled and obtuse-angled) when the lengths of the three sides are given.

The first method is based on Eucl. II. 12 and 13. Let a, b, c be the sides of the triangle opposite to the angles A, B, C respectively, Heron observes (chap. 4) that any angle is acute, right or obtuse according as $c^2 < =$ or $> a^2 + b^2$ is the criterion determining which of the two propositions is applicable. The method is directed to determining the perpendicular segments into which any side is divided by the perpendicular from the opposite vertex.

perpendicular itself. We have, in the cases of the triangle acute-angled at C and the triangle obtuse-angled at C respectively,

$$c^2 = a^2 + b^2 \mp 2a \cdot CD,$$

or
$$CD = \{(a^2 + b^2) \sim c^2\} / 2a,$$



whence $AD^2 (= b^2 - CD^2)$ is found, so that we know the area $(= \frac{1}{2}a \cdot AD)$.

In the cases given in *Metrica* I. 5, 6 the sides are (14, 15, 13) and (11, 13, 20) respectively, and AD is found to be rational ($=12$). But of course both CD (or BD) and AD may be surds, in which case Heron gives approximate values. Cf. *Geom.* 53, 54, Hultsch (15, 1-4, Heib.), where we have a triangle in which $a = 8$, $b = 4$, $c = 6$, so that $a^2 + b^2 - c^2 = 44$ and $CD = 44/16 = 2\frac{1}{2}\frac{1}{4}$. Thus $AD^2 = 16 - (2\frac{1}{2}\frac{1}{4})^2 = 16 - 7\frac{1}{2}\frac{1}{8} = 8\frac{1}{4}\frac{1}{8}\frac{1}{8}$, and $AD = \sqrt{(8\frac{1}{4}\frac{1}{8}\frac{1}{8})} = 2\frac{2}{3}\frac{1}{4}$ approximately, whence the area $= 4 \times 2\frac{2}{3}\frac{1}{4} = 11\frac{2}{3}$. Heron then observes that we get a nearer result still if we multiply AD^2 by $(\frac{1}{2}a)^2$ before extracting the square root, for the area is then $\sqrt{(16 \times 8\frac{1}{4}\frac{1}{8}\frac{1}{8})}$ or $\sqrt{(135)}$, which is very nearly $11\frac{1}{2}\frac{1}{4}\frac{1}{21}$ or $11\frac{1}{2}\frac{3}{4}$.

So in *Metrica* I. 9, where the triangle is 10, 8, 12 (10 being the base), Heron finds the perpendicular to be $\sqrt{63}$, but he obtains the area as $\sqrt{(\frac{1}{4}AD^2 \cdot BC^2)}$, or $\sqrt{(1575)}$, while observing that we can, of course, take the approximation to $\sqrt{63}$, or $7\frac{1}{2}\frac{1}{4}\frac{1}{8}\frac{1}{8}$, and multiply it by half 10, obtaining $39\frac{1}{2}\frac{1}{8}\frac{1}{8}$ as the area.

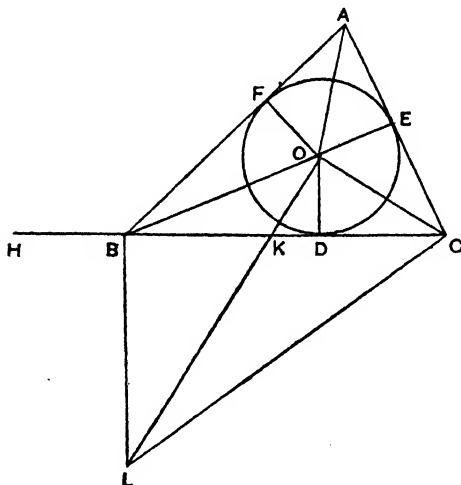
Proof of the formula $\Delta = \sqrt{\{s(s-a)(s-b)(s-c)\}}$.

The second method is that known as the 'formula of Heron', namely, in our notation, $\Delta = \sqrt{\{s(s-a)(s-b)(s-c)\}}$. The proof of the formula is given in *Metrica* I. 8 and also in

Let the sides of the triangle ABC be given in length.
 Inscribe the circle DEF , and let O be the centre.

Join AO, BO, CO, DO, EO, FO .

$$\begin{aligned}\text{Then} \quad BC \cdot OD &= 2 \Delta BOC, \\ CA \cdot OE &= 2 \Delta COA, \\ AB \cdot OF &= 2 \Delta AOB;\end{aligned}$$



whence, by addition,

$$p \cdot OD = 2 \Delta ABC,$$

where p is the perimeter.

Produce CB to H , so that $BH = AF$.

Then, since $AE = AF$, $BF = BD$, and $CE = CD$,
 $CH = \frac{1}{2}p = s$.

$$\text{Therefore} \quad CH \cdot OD = \Delta ABC.$$

But $CH \cdot OD$ is the 'side' of the product $CH^2 \cdot OD^2$,
 $\sqrt{(CH^2 \cdot OD^2)}$,

$$\text{so that} \quad (\Delta ABC)^2 = CH^2 \cdot OD^2.$$

Draw OL at right angles to OC cutting BC in K , and BL at right angles to BC meeting OL in L . Join CL .

Then, since each of the angles COL , CBL is right, $COBL$ is a quadrilateral in a circle.

Therefore $\angle COB + \angle CLB = 2R$.

But $\angle COB + \angle AOF = 2R$, because AO , BO , CO bisect the angles round O , and the angles COB , AOF are together equal to the angles AOC , BOF , while the sum of all four angles is equal to $4R$.

Consequently $\angle AOF = \angle CLB$.

Therefore the right-angled triangles AOF , CLB are similar ;
therefore $BC : BL = AF : FO$

$$= BH : OD,$$

and, alternately, $CB : BH = BL : OD$

$$= BK : KD;$$

whence, *componendo*, $CH : HB = BD : DK$.

It follows that

$$CH^2 : CH \cdot HB = BD \cdot DC : CD \cdot DK$$

$$= BD \cdot DC : OD^2, \text{ since the angle } COK \text{ is right.}$$

Therefore $(\Delta ABC)^2 = CH^2 \cdot OD^2$ (from above)

$$= CH \cdot HB \cdot BD \cdot DC$$

$$= s(s-a)(s-b)(s-c).$$

(β) *Method of approximating to the square root of a non-square number.*

It is à propos of the triangle 7, 8, 9 that Heron gives the important statement of his method of approximating to the value of a surd, which before the discovery of the passage of the *Metrica* had been a subject of unlimited conjecture

the next succeeding square number is 729, which has its side, divide 720 by 27. This gives $26\frac{2}{3}$. Add 27 making $53\frac{2}{3}$, and take half of this or $26\frac{1}{2}\frac{1}{3}$. The side will therefore be very nearly $26\frac{1}{2}\frac{1}{3}$. In fact, if we multiply $26\frac{1}{2}\frac{1}{3}$ by itself, the product is $720\frac{1}{36}$, so that the difference (the square) is $\frac{1}{36}$.

If we desire to make the difference still smaller than $\frac{1}{36}$, we shall take $720\frac{1}{36}$ instead of 729 [or rather we should take $26\frac{1}{2}\frac{1}{3}$ instead of 27], and by proceeding in the same manner we shall find that the resulting difference is much less than $\frac{1}{36}$.

In other words, if we have a non-square number A , the nearest square number to it, so that $A = a^2 \pm b$, we have, as the first approximation to \sqrt{A} ,

$$\alpha_1 = \frac{1}{2} \left(a + \frac{A}{a} \right);$$

for a second approximation we take

$$\alpha_2 = \frac{1}{2} \left(\alpha_1 + \frac{A}{\alpha_1} \right),$$

and so on.²

¹ *Metrica*, i. 8. pp. 18. 22-20. 5.

² The method indicated by Heron was known to Barlaam and Rhabdas in the fourteenth century. The equivalent of it was known to Luca Pacioli (fifteenth-sixteenth century), and it was known to Italian algebraists of the sixteenth century. Thus Luca Pacioli gives $2\frac{1}{2}$, $2\frac{9}{20}$ and $2\frac{881}{1000}$ as successive approximations to $\sqrt{6}$. He gives the first as $2 + \frac{2}{2 \cdot 2}$, the second as $2\frac{1}{2} - \frac{(2\frac{1}{2})^2 - 6}{2 \cdot 2\frac{1}{2}}$, and the third as $2\frac{9}{20} - \frac{(\frac{49}{20})^2 - 6}{2 \cdot \frac{49}{20}}$. The above rule gives $\frac{1}{2}(2 + \frac{6}{2}) = 2\frac{1}{2}$, $\frac{1}{2}(2\frac{1}{2} + \frac{6}{2\frac{1}{2}}) = 2\frac{9}{20}$, and $\frac{1}{2}(2\frac{9}{20} + \frac{6}{2\frac{9}{20}}) = 2\frac{881}{1000}$.

The formula of Heron was again put forward, in modern times, by Buzengeiger as a means of accounting for the Archimedean approximation to $\sqrt{3}$, apparently without knowing its previous history. He also stated it in a treatise on arithmetic (1853). The method which Oppermann and Alexieff sought to account for Archimedean approximations is in reality the same. The latter method deduces the formula

$$\frac{1}{2}(\alpha + \beta) : \sqrt{(\alpha\beta)} = \sqrt{(\alpha\beta)} : \frac{2\alpha\beta}{\alpha + \beta}.$$

Alexieff separated A into two factors a_0, b_0 , and pointed out that

$$a_0 > \sqrt{A} > b_0,$$

then, $\frac{1}{2}(a_0 + b_0) > \sqrt{A} > \frac{2a_0b_0}{a_0 + b_0}$ or $\frac{2a_0b_0}{a_0 + b_0}$,

$$\alpha_1 = a \pm \frac{b}{2a}$$

eron does not seem to have used this formula with a negative sign, unless in *Stereom.* I. 33 (34, Hultsch), where $\sqrt{(63)}$

gain, if $\frac{1}{2}(a_0 + b_0) = a_1$, $2A/(a_0 + b_0) = b_1$,

$$\frac{1}{2}(a_1 + b_1) > \sqrt{A} > \frac{2A}{a_1 + b_1},$$

on.

Now suppose that, in Heron's formulae, we put $a = X_0$, $A/a = x_0$, X_1 , $A/\alpha_1 = x_1$, and so on. We then have

$$a_1 = \frac{1}{2} \left(a + \frac{A}{a} \right) = \frac{1}{2} (X_0 + x_0), \quad x_1 = \frac{A}{X_1} = \frac{A}{\frac{1}{2}(X_0 + x_0)} \text{ or } \frac{2X_0 x_0}{X_0 + x_0};$$

that is, X_1 , x_1 are, respectively, the arithmetic and harmonic means between X_0 , x_0 ; X_2 , x_2 are the arithmetic and harmonic means between X_1 , x_1 , and so on, exactly as in Alexeieff's formulae.

We now try to apply the method to Archimedes's case, $\sqrt{3}$, and we see to what extent it serves to give what we want. Suppose we begin with $3 > \sqrt{3} > 1$. We then have

$$\frac{1}{2}(3+1) > \sqrt{3} > 3 / \frac{1}{2}(3+1), \text{ or } 2 > \sqrt{3} > \frac{2}{3},$$

from this we derive successively

$$\frac{7}{4} > \sqrt{3} > \frac{12}{7}, \quad \frac{97}{56} > \sqrt{3} > \frac{168}{97}, \quad \frac{18817}{10864} > \sqrt{3} > \frac{3251}{18817}.$$

If we start from $\frac{2}{3}$, obtained by the formula $a + \frac{b}{2a+1} < \sqrt{(a^2+b)}$,

we obtain the following approximations by excess,

$$\frac{1}{2} \left(\frac{2}{3} + \frac{3}{2} \right) = \frac{7}{5}, \quad \frac{1}{2} \left(\frac{7}{5} + \frac{5}{7} \right) = \frac{1351}{780}.$$

The second process then gives one of Archimedes's results, $\frac{1351}{780}$, but the other of the two processes gives the other, $\frac{241}{153}$, directly. The latter

may, however, be obtained by using the formula that, if $\frac{a}{b} < \frac{c}{d}$, then

$$\frac{a+nc}{b+nd} < \frac{c}{d}.$$

we can obtain $\frac{241}{153}$ from $\frac{97}{56}$ and $\frac{168}{97}$ thus: $\frac{97+168}{56+97} = \frac{265}{153}$, or from

$\frac{7}{4}$ thus: $\frac{11 \cdot 97 - 7}{11 \cdot 56 - 4} = \frac{1060}{612} = \frac{265}{153}$; and so on. Or again $\frac{1351}{780}$ can

be obtained from $\frac{18817}{10864}$ and $\frac{97}{56}$ thus: $\frac{18817+97}{10864+56} = \frac{18914}{10920} = \frac{1351}{780}$.

The advantage of the method is that, as compared with that of continued fractions, it is a very rapid way of arriving at a close approximation. Günther has shown that the $(m+1)$ th approximation obtained by Heron's formula is the 2^m th obtained by continued fractions. ('Die irrationellen Irrationalitäten der Alten und deren Entwickelungen' in *Abhandlungen zur Gesch. d. Math.* iv. 1882, pp. 83-6.)

is given as approximately $8 - \frac{1}{16}$. In *Metrica* I. 9, as we have seen, $\sqrt{(63)}$ is given as $7\frac{1}{2} \frac{1}{4} \frac{1}{8} \frac{1}{16}$, which was doubtless obtained from the formula (1) as

$$\frac{1}{2}(8 + \frac{63}{8}) = \frac{1}{2}(8 + 7\frac{7}{8}) = 7\frac{1}{2} \frac{1}{4} \frac{1}{8} \frac{1}{16}.$$

The above seems to be the only *classical* rule which has been handed down for finding second and further approximations to the value of a surd. But, although Heron shows how to obtain a second approximation, namely by formula (2), he does not seem to make any direct use of this method himself, and consequently the question how many approximations closer than the first which are to be found in his works were obtained still remains an open one.

(γ) *Quadrilaterals.*

It is unnecessary to give in detail the methods of measuring the areas of quadrilaterals (chaps. 11–16). Heron deals with the following kinds, the parallel-trapezium (isosceles or scalene), the rhombus and rhomboid, and the quadrilateral which has one angle right and in which the four sides are of given lengths. Heron points out that in the rhombus and rhomboid, and in the general case of the quadrilateral, it is necessary to know a diagonal as well as the four sides. In mensuration in all the cases reduces to that of the rectangle and triangle.

(δ) *The regular polygons with 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12 sides.*

Beginning with the *equilateral triangle* (chap. 17), Heron proves that, if a be the side and p the perpendicular from a vertex on the opposite side, $a^2 : p^2 = 4 : 3$, whence

$$a^4 : p^2 a^2 = 4 : 3 = 16 : 12,$$

so that

$$a^4 : (\Delta ABC)^2 = 16 : 3,$$

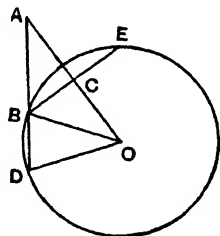
and $(\Delta ABC)^2 = \frac{3}{16} a^4$. In the particular case taken $a =$

Heron for $\sqrt{5}$ is here $\frac{13}{15}$. For the side 10, the method gives the same result as above, for $\frac{13}{15} \cdot 100 = 43\frac{1}{3}$.

The regular *pentagon* is next taken (chap. 18). Heron premises the following lemma.

Let ABC be a right-angled triangle, with the angle A equal to $\frac{2}{5}R$. Produce AC to O so that $CO = AC$.

If now AO is divided in extreme and mean ratio, AB is equal to the greater segment. (For produce AB to D so that $AD = AO$, and join BO , DO . Then, since ADO is isosceles and the angle at $A = \frac{2}{5}R$, $\angle ADO = \angle AOD = \frac{4}{5}R$, and, from the equality of the triangles ABC , OBC , $\angle AOB = \angle BAO = \frac{2}{5}R$. It follows that



the triangle ADO is the isosceles triangle of Eucl. IV. 10, and AD is divided in extreme and mean ratio in B .) Therefore, says Heron, $(BA + AC)^2 = 5 AC^2$. [This is Eucl. XIII. 1.]

Now, since $\angle BOC = \frac{2}{5}R$, if BC be produced to E so that $CE = BC$, BE subtends at O an angle equal to $\frac{4}{5}R$, and therefore BE is the side of a regular pentagon inscribed in the circle with O as centre and OB as radius. (This circle also passes through D , and BD is the side of a regular decagon in the same circle.) If now $BO = AB = r$, $OC = p$, $BE = a$, we have from above, $(r + p)^2 = 5p^2$, whence, since $\sqrt{5}$ is approximately $\frac{9}{4}$, we obtain approximately $r = \frac{5}{4}p$, and $\frac{1}{2}a = \frac{3}{4}p$, so that $p = \frac{2}{3}a$. Hence $\frac{1}{2}pa = \frac{1}{3}a^2$, and the area of the pentagon $= \frac{5}{2}a^2$. Heron adds that, if we take a closer approximation to $\sqrt{5}$ than $\frac{9}{4}$, we shall obtain the area still more exactly. In the *Geometry*¹ the formula is given as $\frac{1}{7}a^2$.

The regular *hexagon* (chap. 19) is simply 6 times the equilateral triangle with the same side. If Δ be the area of the equilateral triangle with side a , Heron has proved that $\Delta^2 = \frac{3}{16}a^4$ (*Metrica* I. 17), hence $(\text{hexagon})^2 = \frac{27}{4}a^4$. If, e.g. $a = 10$, $(\text{hexagon})^2 = 67500$, and $(\text{hexagon}) = 259$ nearly. In the *Geometry*² the formula is given as $\frac{1}{5}a^2$, while 'another book' is quoted as giving $6(\frac{1}{3} + \frac{1}{16})a^2$; it is added that the latter formula, obtained from the area of the triangle, $(\frac{1}{3} + \frac{1}{16})a^2$, represents the more accurate procedure, and is fully set out by

¹ *Geom.* 102 (21, 14, Heib.).

² *Ib.* 102 (21, 16, 17, Heib.).

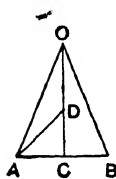
Heron. As a matter of fact, however, $6(\frac{1}{3} + \frac{1}{10}) = \frac{1}{5}$ exactly, and only the *Metrica* gives the more accurate calculation.

The regular *heptagon*.

Heron assumes (chap. 20) that, if a be the side and r the radius of the circumscribing circle, $a = \frac{7}{8}r$, being approximately equal to the perpendicular from the centre of the circle to the side of the regular hexagon inscribed in it (for $\frac{7}{8}$ is the approximate value of $\frac{1}{2}\sqrt{3}$). This theorem is quoted by Jordanus Nemorarius (d. 1237) as an 'Indian rule'; he probably obtained it from Abū'l Wafā (940-98). The *Metrica* shows that it is of Greek origin, and, if Archimedes really wrote a book on the heptagon in a circle, it may be due to him. If then p is the perpendicular from the centre of the circle on the side (a) of the inscribed heptagon, $r/(\frac{1}{2}a) = 8/3$, or $16/7$, whence $p^2/(\frac{1}{2}a)^2 = \frac{207}{49}$, and $p/\frac{1}{2}a =$ (approximately) $14\frac{1}{3}/7$ or $43/21$. Consequently the area of the heptagon $= 7 \cdot \frac{1}{2}pa = 7 \cdot \frac{43}{21} \frac{a^2}{2} = \frac{43}{6}a^2$.

The regular *octagon*, *decagon* and *dodecagon*.

In these cases (chaps. 21, 23, 25) Heron finds p by drawing the perpendicular OC from O , the centre of the circumscribed circle, on a side AB , and then making the angle OAD equal to the angle AOD .



For the octagon,

$$\angle ADC = \frac{1}{2}R, \text{ and } p = \frac{1}{2}a(1 + \sqrt{2}) = \frac{1}{2}a(1 + \frac{1}{2}) \text{ or } \frac{1}{2}a \cdot \frac{3}{2} \text{ approximately.}$$

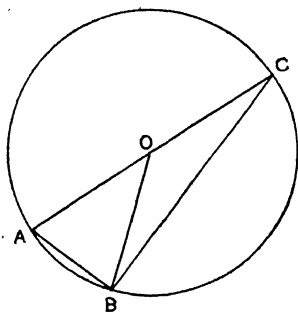
For the decagon,

$$\angle ADC = \frac{2}{5}R, \text{ and } AD:DC = 5:4 \text{ nearly (see preceding page) hence } AD:AC = 5:3, \text{ and } p = \frac{1}{2}a(\frac{5}{3} + \frac{4}{3}) = \frac{3}{2}a.$$

For the dodecagon,

$$\angle ADC = \frac{1}{3}R, \text{ and } p = \frac{1}{2}a(2 + \sqrt{3}) = \frac{1}{2}a(2 + \frac{7}{4}) = \frac{15}{8}a \text{ approximately.}$$

presumably Hipparchus's Table) is appealed to. If AB be the side (a) of an enneagon or hendecagon inscribed in a circle, AC the diameter through A , we are told that the Table of Chords gives $\frac{1}{3}$ and $\frac{7}{25}$ as the respective approximate values of the ratio AB/AC . The angles subtended at the centre O by the side AB are 40° and $32\frac{8}{11}^\circ$ respectively, and Ptolemy's Table gives, as the chords subtended by angles of 40° and 33° respectively, $41^p 2' 33''$ and $34^p 4' 55''$ (expressed in 120th parts of the diameter); Heron's figures correspond to 40^p and $33^p 36'$ respectively. For the *enneagon* $AC^2 = 9AB^2$, whence $BC^2 = 8AB^2$ or approximately $\frac{239}{8}AB^2$, and $BC = \frac{17}{6}a$; therefore (area of enneagon) $= \frac{9}{2} \cdot \Delta ABC = \frac{51}{8}a^2$. For the *hendecagon* $AC^2 = \frac{625}{49}AB^2$ and $BC^2 = \frac{576}{49}AB^2$, so that $BC = \frac{24}{7}a$, and area of hendecagon $= \frac{11}{2} \cdot \Delta ABC = \frac{66}{7}a^2$.



An ancient formula for the ratio between the side of any regular polygon and the diameter of the circumscribing circle is preserved in *Geëpon*. 147 sq. (= Pseudo-Dioph. 23-41), namely $d_n = n \frac{a_n}{3}$. Now the ratio na_n/d_n tends to π as the number (n) of sides increases, and the formula indicates a time when π was generally taken as ≈ 3 .

(ε) *The Circle.*

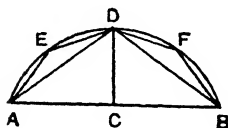
Coming to the circle (*Metrical* I. 26) Heron uses Archimedes's value for π , namely $\frac{22}{7}$, making the circumference of a circle $\frac{44}{7}r$ and the area $\frac{1}{2} \frac{1}{4} d^2$, where r is the radius and d the diameter. It is here that he gives the more exact limits for π which he says that Archimedes found in his work *On Plinthis and Cylinders*, but which are not convenient for calculations. The limits, as we have seen, are given in the text as $\frac{211875}{67441} < \pi < \frac{197888}{62351}$, and with Tannery's alteration to $\frac{211872}{67441} < \pi < \frac{195882}{62351}$ are quite satisfactory.¹

¹ See vol. i, pp. 232-3.

(5) *Segment of a circle.*

According to Heron (*Metrica* I. 30) the ancients measured the area of a segment rather inaccurately, taking it to be $\frac{1}{2}(b+h)h$, where b is the base and h the height. Heron conjectures that it arose from taking $\pi = 3$, because if we apply the formula to the semicircle, the area becomes $\frac{1}{2}(2r+r)r = \frac{3}{2}r^2$, where r is the radius. Those, he says (chap. 31), who have investigated the area more accurately have added $\frac{1}{14}$ to the above formula, making it $\frac{1}{2}(b+h)h + \frac{1}{14}(\frac{1}{2}b)^2$, which seems to correspond to the value $3\frac{1}{7}$ for π , since, when applied to the semicircle, the formula gives $\frac{1}{2}(3r^2 + \frac{1}{7}r^2)$. Heron thinks that this formula should only be applied to segments of a circle less than a semicircle, and not even to all of them, only in cases where b is not greater than $3h$. Suppose that $b = 60$, $h = 1$; in that case even $\frac{1}{14}(\frac{1}{2}b)^2 = \frac{1}{14} \cdot 900$, which is greater even than the parallelogram with sides b and h , which again is greater than the segment. Wherefore $b > 3h$, he adopts another procedure.

This is exactly modelled on Archimedes's quadrature of a segment of a parabola. Heron proves (*Metrica* I. 29) that, if ADB be a segment of a circle, and D the midpoint of the arc, and if the arcs AD and BD are similarly bisected at E , F ,



$$\Delta ADB < 4(\Delta AED + \Delta BFD)$$

Similarly, if the same construction is made for the segments AED , BFD , the area of each of them is less than 4 times the sum of the two small segments. It follows that

$$\begin{aligned} (\text{area of segmt. } ADB) &> \Delta ADB \left\{ 1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \dots \right\} \\ &> \frac{4}{3} \Delta ADB. \end{aligned}$$

'If therefore we measure the triangle, and add one-fourth of it, we shall obtain the area of the segment as nearly as possible.' That is, for segments in which $b > 3h$, Heron takes the area to be equal to that of the parabolic segment with the same base and height, or $\frac{2}{3}bh$.

than a semicircle, the second to a segment less than a semicircle.

In the *Metrica* the area of a segment greater than a semicircle is obtained by subtracting the area of the complementary segment from the area of the circle.

From the *Geometrica*¹ we find that the circumference of the segment less than a semicircle was taken to be $\sqrt{(b^2 + 4h^2)} + \frac{1}{4}h$ or alternatively $\sqrt{(b^2 + 4h^2)} + \{\sqrt{(b^2 + 4h^2)} - b\} \frac{h}{b}$.

(η) *Ellipse, parabolic segment, surface of cylinder, right cone, sphere and segment of sphere.*

After the area of an ellipse (*Metrica* I. 34) and of a parabolic segment (chap. 35), Heron gives the surface of a cylinder (chap. 36) and a right cone (chap. 37); in both cases he unrolls the surface on a plane so that the surface becomes that of a parallelogram in the one case and a sector of a circle in the other. For the surface of a sphere (chap. 38) and a segment of it (chap. 39) he simply uses Archimedes's results.

Book I ends with a hint how to measure irregular figures, plane or not. If the figure is plane and bounded by an irregular curve, neighbouring points are taken on the curve such that, if they are joined in order, the contour of the polygon so formed is not much different from the curve itself, and the polygon is then measured by dividing it into triangles. If the surface of an irregular solid figure is to be found, you wrap round it pieces of very thin paper or cloth, enough to cover it, and you then spread out the paper or cloth and measure that.

Book II. Measurement of volumes.

The preface to Book II is interesting as showing how vague the traditions about Archimedes had already become.

'After the measurement of surfaces, rectilinear or not, it is proper to proceed to the solid bodies, the surfaces of which we have already measured in the preceding book, surfaces plane and spherical, conical and cylindrical, and irregular surfaces as well. The methods of dealing with these solids are, in

¹ Cf. *Geom.*, 94, 95 (19. 2, 4, Heib.), 97. 4 (20. 7, Heib.).

view of their surprising character, referred to Archimedes by certain writers who give the traditional account of their origin. But whether they belong to Archimedes or another, it is necessary to give a sketch of these methods as well.'

The Book begins with generalities about figures all the sections of which parallel to the base are equal to the base and similarly situated, while the centres of the sections are on a straight line through the centre of the base, which may be either obliquely inclined or perpendicular to the base; whether the said straight line ('the axis') is or is not perpendicular to the base, the volume is equal to the product of the area of the base and the *perpendicular* height of the top of the figure from the base. The term 'height' is thenceforward restricted to the length of the perpendicular from the top of the figure on the base.

(a) *Cone, cylinder, parallelepiped (prism), pyramid, and frustum.*

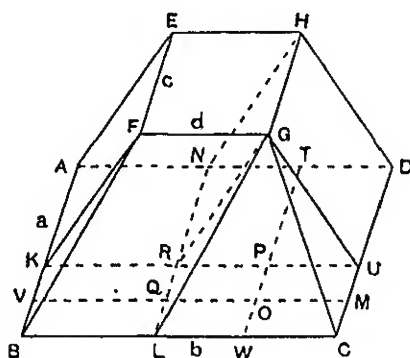
II. 1-7 deal with a cone, a cylinder, a 'parallelepiped' (the base of which is not restricted to the parallelogram but is in the illustration given a regular hexagon, so that the figure is more properly a prism with polygonal bases), a triangular prism, a pyramid with base of any form, a frustum of a triangular pyramid; the figures are in general *oblique*.

(β) *Wedge-shaped solid (βωμίσκος or σφηνίσκος).*

II. 8 is a case which is perhaps worth giving. It is that of a rectilinear solid, the base of which is a rectangle $ABCD$ and has opposite to it another rectangle $EFGH$, the sides of which are respectively parallel but not necessarily proportional to those of $ABCD$. Take AK equal to EF , and BL equal to FG . Bisect BK , CL in V , W , and draw $KRPV$, $VQOM$ parallel to AD , and $LQRN$, $WOPT$ parallel to AB . Join FK , GR , LG , GU , HN .

Then the solid is divided into (1) the parallelepiped with AR , EG as opposite faces, (2) the prism with KL as base and FG as the opposite edge, (3) the prism with NU as base and GH as opposite edge, and (4) the pyramid with $RLCU$ as base and G as vertex. Let h be the 'height' of the figure. Now

the parallelepiped (1) is on AR as base and has height h ; the prism (2) is equal to a parallelepiped on KQ as base and with height h ; the prism (3) is equal to a parallelepiped with NP as base and height h ; and finally the pyramid (4) is equal to a parallelepiped of height h and one-third of RC as base.



Therefore the whole solid is equal to one parallelepiped with height h and base equal to $(AR + KQ + NP + RO + \frac{1}{3}RO)$ or $AO + \frac{1}{3}RO$.

Now, if $AB = a$, $BC = b$, $EF = c$, $FG = d$,

$AV = \frac{1}{2}(a + c)$, $AT = \frac{1}{2}(b + d)$, $RQ = \frac{1}{2}(a - c)$, $RP = \frac{1}{2}(b - d)$.

Therefore volume of solid

$$= \left\{ \frac{1}{4}(a + c)(b + d) + \frac{1}{12}(a - c)(b - d) \right\} h.$$

The solid in question is evidently the true $\beta\omega\mu\sigma\kappa\omicron\varsigma$ ('little star'), for the formula is used to calculate the content of $\beta\omega\mu\sigma\kappa\omicron\varsigma$ in *Stereom.* II. 40 (68, Heib.) It is also, I think, the $\sigma\phi\eta\nu\sigma\kappa\omicron\varsigma$ ('little wedge'), a measurement of which is given in *Stereom.* I. 26 (25, Heib.) It is true that the second term of the first factor $\frac{1}{12}(a - c)(b - d)$ is there neglected, perhaps because in the case taken ($a = 7$, $b = 6$, $c = 5$, $d = 4$) this term ($= \frac{1}{3}$) is small compared with the other (≈ 30). A particular $\sigma\phi\eta\nu\sigma\kappa\omicron\varsigma$, in which either $c = a$ or $d = b$, was called $\delta\nu\nu\acute{\xi}$; the second term in the factor of the content vanishes in this case, and, if e.g. $c = a$, the content is $\frac{1}{4}(b + d)ah$. Another $\beta\omega\mu\sigma\kappa\omicron\varsigma$ is measured in *Stereom.* I. 35 (34, Heib.), where the solid is inaccurately called 'a pyramid oblong $\acute{\epsilon}\tau\epsilon\rho\omicron\mu\acute{\eta}\kappa\eta\varsigma$) and truncated ($\kappa\acute{\omicron}\lambda\omicron\upsilon\pi\omicron\varsigma$) or half-perfect'.

The method is the same *mutatis mutandis* as that used in II. 6 for the frustum of a pyramid with any triangle for base, and it is applied in II. 9 to the case of a frustum of a pyramid with a square base, the formula for which is

$$\left[\left\{\frac{1}{2}(a+a')\right\}^2 + \frac{1}{3}\left\{\frac{1}{2}(a-a')\right\}^2\right]h,$$

where a, a' are the sides of the larger and smaller bases respectively, and h the height; the expression is of course easily reduced to $\frac{1}{3}h(a^2 + aa' + a'^2)$.

(γ) *Frustum of cone, sphere, and segment thereof.*

A *frustum of a cone* is next measured in two ways, (1) by comparison with the corresponding frustum of the circumscribing pyramid with square base, (2) directly as the difference between two cones (chaps. 9, 10). The volume of the frustum of the cone is to that of the frustum of the circumscribing pyramid as the area of the base of the cone to that of the base of the pyramid; i.e. the volume of the frustum of the cone is $\frac{1}{2}\pi$, or $\frac{1}{2}\frac{1}{2}$, times the above expression for the frustum of the pyramid with a^2, a'^2 as bases, and it reduces to $\frac{1}{2}\pi h(a^2 + aa' + a'^2)$, where a, a' are the *diameters* of the two bases. For the *sphere* (chap. 11) Heron uses Archimedes's proposition that the circumscribing cylinder is $1\frac{1}{2}$ times the sphere, whence the volume of the sphere = $\frac{2}{3}d \cdot \frac{1}{2}d^2$ or $\frac{1}{6}d^3$; for a *segment of a sphere* (chap. 12) he likewise uses Archimedes's result (*On the Sphere and Cylinder*, II. 4).

(δ) *Anchor-ring or tore.*

The anchor-ring or *tore* is next measured (chap. 13) by means of a proposition which Heron quotes from Dionysodorus, and which is to the effect that, if a be the radius of either circular section of the *tore* through the axis of revolution, and c the distance of its centre from that axis,

$$\pi a^2 : ac = (\text{volume of tore}) : \pi c^2 \cdot 2a$$

[whence volume of tore = $2\pi^2 ca^2$]. In the particular case taken $a = 6$, $c = 14$, and Heron obtains, from the proportion $113\frac{1}{7} : 84 = V : 7392$, $V = 9956\frac{4}{7}$. But he shows that he is aware that the volume is the product of the area of the

cribing circle and the length of the path of its centre. He says, since 14 is a radius (of the path of the centre), its diameter and 88 its circumference. 'If then the tore straightened out and made into a cylinder, it will have 88 its length, and the diameter of the base of the cylinder is so that the solid content of the cylinder is, as we have 9956 $\frac{1}{2}$ ' ($= 88 \cdot \frac{1}{2} \cdot 144$).

(ε) *The two special solids of Archimedes's 'Method'.*

Chaps. 14, 15 give the measurement of the two remarkable solids of Archimedes's *Method*, following Archimedes's results.

(ζ) *The five regular solids.*

Chaps. 16-18 Heron measures the content of the five regular solids after the cube. He has of course in each case the perpendicular from the centre of the circumscribed sphere on any face. Let p be this perpendicular, a the side of the solid, r the radius of the circle circumscribing any face. Then (1) for the *tetrahedron*

$$a^2 = 3r^2, p^2 = a^2 - \frac{1}{3}a^2 = \frac{2}{3}a^2.$$

In the case of the *octahedron*, which is the sum of two square pyramids on a square base, the content is one-third that base multiplied by the diagonal of the figure, $\frac{1}{3}a^2 \cdot \sqrt{2}a$ or $\frac{1}{3}\sqrt{2} \cdot a^3$; in the case taken $a = 7$, and Heron takes 10 as an approximation to $\sqrt{(2 \cdot 7^2)}$ or $\sqrt{98}$, the result being $\frac{1}{3} \cdot 10 \cdot 49$ or $163\frac{1}{3}$. (3) In the case of the *icosaedron* Heron merely says that

$$a = 93 : 127 \left(\text{the real value of the ratio is } \frac{1}{2} \sqrt{\frac{7+3\sqrt{5}}{6}} \right).$$

In the case of the *dodecahedron*, Heron says that $a = 9 : 8$ (the true value is $\frac{1}{2} \sqrt{\frac{25+11\sqrt{5}}{10}}$, and, if $\sqrt{5}$ is taken equal to $\frac{9}{4}$, Heron's ratio is readily obtained).

Book II ends with an allusion to the method attributed to Archimedes for measuring the contents of irregular bodies by immersing them in water and measuring the amount of fluid displaced.

Book III. Divisions of figures.

This book has much in common with Euclid's book *On divisions (of figures)*, the problem being to divide various figures, plane or solid, by a straight line or plane into parts having a given ratio. In III. 1-3 a triangle is divided into two parts in a given ratio by a straight line (1) passing through a vertex, (2) parallel to a side, (3) through any point on a side. III. 4 is worth description: 'Given a triangle ABC , to cut out of it a triangle DEF (where D, E, F are points on the sides respectively) given in magnitude and such that the triangles AEF, BFD, CED may be equal in area.' Heron assumes that, if D, E, F divide the sides so that

$$AF:FB = BD:DC = CE:EA,$$

the latter three triangles are equal in area.

He then has to find the value of each of the three ratios which will result in the triangle DEF having a given area.

Join AD .

Since $BD:CD = CE:EA$,

$$BC:CD = CA:AE,$$

and $\triangle ABC:\triangle ADC = \triangle ADC:\triangle ADE$.

Also $\triangle ABC:\triangle ABD = \triangle ADC:\triangle EDC$.

But (since the area of the triangle DEF is given) $\triangle EDC$ is given, as well as $\triangle ABC$. Therefore $\triangle ABD \times \triangle ADC$ is given.

Therefore, if AH be perpendicular to BC ,

$AH^2 \cdot BD \cdot DC$ is given;

therefore $BD \cdot DC$ is given, and, since BC is given, D is given in position (we have to apply to BC a rectangle equal to $BD \cdot DC$ and falling short by a square).

As an example Heron takes $AB = 13$, $BC = 14$, $CA = 16$, $\triangle DEF = 24$. $\triangle ABC$ is then 84, and $AH = 12$.

Thus $\triangle EDC = 20$, and $AH^2 \cdot BD \cdot DC = 4 \cdot 84 \cdot 20 = 6720$; therefore $BD \cdot DC = 6720/144$ or $46\frac{2}{3}$ (the text omits the $\frac{2}{3}$).

Therefore, says Heron, $BD = 8$ approximately. For 8 we

could apparently have $8\frac{1}{2}$, since DC is immediately stated to be $5\frac{1}{2}$ (not 6). That is, in solving the equation

$$x^2 - 14x + 46\frac{2}{3} = 0,$$

which gives $x = 7 \pm \sqrt{(2\frac{1}{3})}$, Heron apparently substituted $2\frac{1}{4}$ or for $2\frac{1}{3}$, thereby obtaining $1\frac{1}{4}$ as an approximation to the root.

(The lemma assumed in this proposition is easily proved. Let $m : n$ be the ratio $AF : FB = BD : DC = CE : EA$.

Then $AF = mo/(m+n)$, $FB = nc/(m+n)$, $CE = mb/(m+n)$,

$$EA = nb/(m+n), \text{ \&c.}$$

hence

$$\triangle AFE/\triangle ABC = \frac{mn}{(m+n)^2} = \triangle BDF/\triangle ABC = \triangle CDE/\triangle ABC,$$

and the triangles AFE , BDF , CDE are equal.

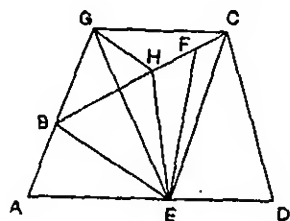
Pappus¹ has the proposition that the triangles ABC , DEF have the same centre of gravity.)

Heron next shows how to divide a parallel-trapezium into two parts in a given ratio by a straight line (1) through the point of intersection of the non-parallel sides, (2) through a given point on one of the parallel sides, (3) parallel to the parallel sides, (4) through a point on one of the non-parallel sides (III. 5-8). III. 9 shows how to divide the area of a circle into parts which have a given ratio by means of an inner circle with the same centre. For the problems beginning with III. 10 Heron says that numerical calculation alone no longer suffices, but geometrical methods must be applied. These problems are reduced to problems solved by Apollonius in his treatise *On cutting off an area*. The first of these is III. 10, to cut off from the angle of a triangle a given portion of the triangle by a straight line through a point on the opposite side produced. III. 11, 12, 13 show how to cut any quadrilateral into parts in a given ratio by a straight line through a point (1) on a side (a) dividing the area in the given ratio, (b) not so dividing it, (2) not on any side, (a) in the case where the quadrilateral is a trapezium, (b) in the case where it is not; the case (b) is reduced (like III. 10) to the 'cutting-off' of an

¹ Pappus, viii, pp. 1034-8. Cf. pp. 430-2 *post*.

area'. These propositions are ingenious and interesting. III. 11 shall be given as a specimen.

Given any quadrilateral $ABCD$ and a point E on the side AD , to draw through E a straight line EF which shall cut the quadrilateral into two parts in the ratio of AE to ED . (We omit the analysis.) Draw CG parallel to DA to meet AB produced in G .



Join BE , and draw GH parallel to BE meeting BC in H .

Join CE , EH , EG .

Then $\triangle GBE = \triangle HBE$ and, adding $\triangle ABE$ to each, we have

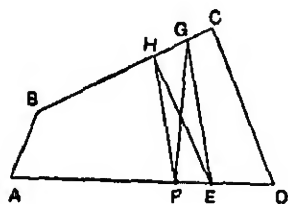
$$\triangle AGE = (\text{quadrilateral } ABHE).$$

$$\begin{aligned} \text{Therefore (quadr. } ABHE) : \triangle CED &= \triangle AGE : \triangle CED \\ &= AE : ED. \end{aligned}$$

But (quadr. $ABHE$) and $\triangle CED$ are parts of the quadrilateral, and they leave over only the triangle EHC . We have therefore only to divide $\triangle EHC$ in the same ratio $AE : ED$ by the straight line EF . This is done by dividing HC at F in the ratio $AE : ED$ and joining EF .

The next proposition (III. 12) is easily reduced to this.

If $AE : ED$ is not equal to the given ratio, let F divide AD in the given ratio, and through F draw FG dividing the quadrilateral in the given ratio (III. 11).



Join EG , and draw FH parallel to EG . Let FH meet BC in H , and join EH .

Then is EH the required straight line through E dividing the quadrilateral in the given ratio.

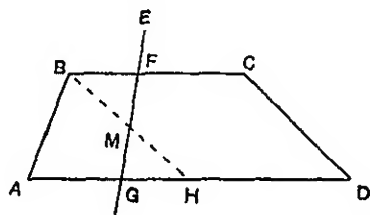
For $\triangle FGE = \triangle HGE$. Add to each (quadr. $GEDC$).

Therefore (quadr. $CGFD$) = (quadr. $CHED$).

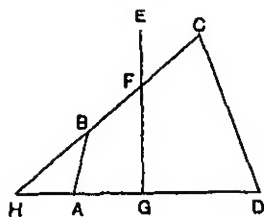
Therefore EH divides the quadrilateral in the given ratio, just as FG does.

The case (III. 13) where E is not on a side of the quadrilateral [(2) above] takes two different forms according as the

opposite sides which the required straight line cuts are parallel or (b) not parallel. In the first case (a) the problem reduces to drawing a straight line through E intersecting the parallel sides in points F, G such that $BF + AG$



equal to a given length. In the second case (b) where BC, AD are not parallel Heron supposes them to meet in H . The angle at H is then given, and the area ABH . It is then a question of cutting off from a triangle with vertex H a triangle HFG of given area by a straight line drawn from E , which is again a problem in Apollonius's *Cutting-off of an*

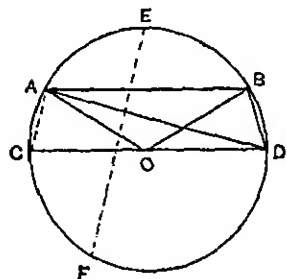


16. The auxiliary problem in case (a) is easily solved in II. 16. Measure AH equal to the given length. Join BH and bisect it at M . Then EM meets BC, AD in points such that $BF + AG =$ the given length. For, by congruent triangles, $BF = GI$.

The same problems are solved for the case of any polygon in II. 14, 15. A sphere is then divided (III. 17) into segments such that their surfaces are in a given ratio, by means of Archimedes, *On the Sphere and Cylinder*, II. 3, just as, in II. 23, Prop. 4 of the same Book is used to divide a sphere into segments having their volumes in a given ratio.

III. 18 is interesting because it recalls an ingenious proposition in Euclid's book *On Divisions*. Heron's problem is to divide a given circle into three equal parts by two straight

lines', and he observes that, 'as the problem is clearly not rational, we shall, for practical convenience, make the division, as exactly as possible, in the following way.' AB is the side of an equilateral triangle inscribed in the circle. Let CD be the parallel diameter, O the centre of the circle, and join AO , BO , AD , DB . Then shall the segment ABD be very nearly one-third of the circle. For,



since AB is the side of an equilateral triangle in the circle, the sector $OAEB$ is one-third of the

circle. And the triangle AOR forming part of the sector is equal to the triangle ADB ; therefore the segment AEB plus the triangle ABD is equal to one-third of the circle, and the segment ABD only differs from this by the small segment on BD as base, which may be neglected. Euclid's proposition is to cut off one-third (or any fraction) of a circle between two parallel chords (see vol. i, pp. 429-30).

III. 19 finds a point D within any triangle ABC such that the triangles DBC , DCA , DAB are all equal; and then Heron passes to the division of solid figures.

The solid figures divided in a given ratio (besides the sphere) are the pyramid with base of any form (III. 20), the cone (III. 21) and the frustum of a cone (III. 22), the cutting planes being parallel to the base in each case. These problems involve the extraction of the cube root of a number which is in general not an exact cube, and the point of interest is Heron's method of approximating to the cube root in such a case. Take the case of the cone, and suppose that the portion to be cut off at the top is to the rest of the cone as m to n . We have to find the ratio in which the height or the edge is cut by the plane parallel to the base which cuts the cone in the given ratio. The volume of a cone being $\frac{1}{3}\pi c^2h$, where c is the radius of the base and h the height, we have to find the height of the cone the volume of which is $\frac{m}{m+n} \cdot \frac{1}{3}\pi c^2h$, and, as the height h' is to the radius c' of its base as h is to c , we have simply to find h' where

$h^3 = m/(m+n)$. Or, if we take the edges e, e' instead of the heights, $e'^3/e^3 = m/(m+n)$. In the case taken by Heron $m:n = 4:1$, and $e = 5$. Consequently $e'^3 = \frac{4}{5} \cdot 5^3 = 100$. Therefore, says Heron, $e' = \sqrt[3]{100}$ approximately, and in III. 20 shows how this is arrived at.

Approximation to the cube root of a non-cube number.

Take the nearest cube numbers to 100 both above and below; these are 125 and 64.

$$\text{Then } 125 - 100 = 25,$$

$$100 - 64 = 36.$$

Multiply 5 into 36; this gives 180. Add 100, making 280. Divide 180 by 280; this gives $\frac{9}{14}$. Add this to the side of smaller cube: this gives $4\frac{9}{14}$. This is as nearly as possible the cube root ("cubic side") of 100 units.

We have to conjecture Heron's formula from this example. Generally, if $a^3 < A < (a+1)^3$, suppose that $A - a^3 = d_1$, and $(a+1)^3 - A = d_2$. The best suggestion that has been made by Wertheim's,¹ namely that Heron's formula for the approxi-

mate cube root was $a + \frac{(a+1)d_1}{(a+1)d_1 + ad_2}$. The 5 multiplied

by the 36 might indeed have been the square root of 25 or d_1 , and the 100 added to the 180 in the denominator of the fraction might have been the original number 100 (A) and not 5 or ad_2 , but Wertheim's conjecture is the more satisfactory because it can be evolved out of quite elementary considerations. This is shown by G. Eneström as follows.² Using the same notation, Eneström further supposes that x is the exact cube root of $\sqrt[3]{A}$, and that $(x-a)^3 = \delta_1$, $(a+1-x)^3 = \delta_2$.

Thus

$$a^3 - 3x^2a + 3xa^2 - a^3, \text{ and } 3ax(x-a) = a^3 - a^3 - \delta_1 = d_1 - \delta_1.$$

Similarly from $\delta_2 = (a+1-x)^3$ we derive

$$3(a+1)x(a+1-x) = (a+1)^3 - a^3 - \delta_2 = d_2 - \delta_2.$$

Therefore

$$\begin{aligned} \frac{\delta_2}{\delta_1} &= \frac{3(a+1)x(a+1-x)}{3ax(x-a)} = \frac{(a+1)\{1-(x-a)\}}{a(x-a)} \\ &= \frac{a+1}{a(x-a)} - \frac{a+1}{a}; \end{aligned}$$

¹ *Zeitschr. f. Math. u. Physik*, xlv, 1899, hist.-litt. Abt., pp. 1-3.

² *Bibliotheca Mathematica*, viii, 1907-8, pp. 412-13.

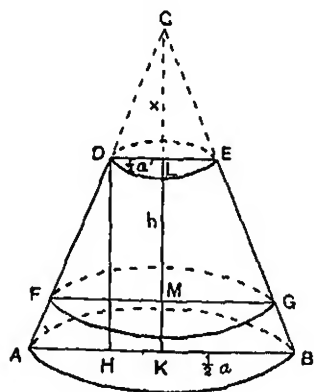
and, solving for $x - a$, we obtain

$$x - a = \frac{(a+1)(d_1 - \delta_1)}{(a+1)(d_1 - \delta_1) + a(d_2 - \delta_2)},$$

or
$$\sqrt[3]{A} = a + \frac{(a+1)(d_1 - \delta_1)}{(a+1)(d_1 - \delta_1) + a(d_2 - \delta_2)}.$$

Since δ_1, δ_2 are in any case the cubes of fractions, we may neglect them for a first approximation, and we have

$$\sqrt[3]{A} = a + \frac{(a+1)d_1}{(a+1)d_1 + ad_2}.$$



III. 22, which shows how to cut a frustum of a cone in a given ratio by a section parallel to the bases, shall end our account of the *Metrica*. I shall give the general formulæ on the left and Heron's case on the right. Let $ABED$ be the frustum, let the *diameters* of the bases be a, a' , and the height h . Complete the cone, and let the height of CDE be x .

Suppose that the frustum has to be cut by a plane FG in such a way that

$$(\text{frustum } DG) : (\text{frustum } FB) = m : n.$$

In the case taken by Heron

$$a = 28, a' = 21, h = 12, m = 4, n = 1.$$

Draw DH perpendicular to AB .

$$(DG):(FB) = m:n,$$

$$(DB):(DG) = (m+n):m.$$

$$= \frac{1}{12} \pi h (a^2 + aa' + a'^2),$$

$$(DG) = \frac{m}{m+n} (DB).$$

be the height (CM) of the
 G .

$$DH:AH = CK:KA,$$

$$(a-a') = (x+h):\frac{1}{2}a,$$

is known.

$$CDE = \frac{1}{12} \pi a'^2 x,$$

$$CFG = (CDE) + \frac{m}{m+n} (DB),$$

$$CAB = (CDE) + (DB).$$

says Heron,

$$\frac{(CFG)}{(CDE)} = \frac{(x+h)^3 + x^3}{y^3}.$$

might have said simply

$$(CDE):(CFG) = x^3:y^3.]$$

gives y or CM ,

LM is known.

$$AD^2 = AH^2 + DH^2$$

$$= \left\{ \frac{1}{2}(a-a') \right\}^2 + h^2,$$

AD is known.

$$\text{Hence } DF = \frac{y-x}{h} \cdot AD \text{ is}$$

$$(DG):(FB) = 4:1,$$

$$(DB):(DG) = 5:4.$$

$$(DB) = 5698,$$

$$(DG) = 4558\frac{2}{5}.$$

$$x+h = \frac{14 \cdot 12}{3\frac{1}{2}} = 48,$$

$$\text{and } x = 48 - 12 = 36.$$

$$(\text{cone } CDE) = 4158,$$

$$(\text{cone } CFG) = 4158 + 4558\frac{2}{5} = 8716\frac{2}{5},$$

$$(\text{cone } CAB) = 4158 + 5698 = 9856.$$

$$y^3 = \frac{8716\frac{2}{5}}{9856 + 4158} \cdot (48^3 + 36^3)$$

$$= 8716\frac{2}{5} \cdot \frac{157248}{14614} = 97806,$$

whence $y = 46$ approximately.

$$\text{Therefore } LM = y - x = 10.$$

$$AD^2 = (3\frac{1}{2})^2 + 12^2$$

$$= 156\frac{1}{4},$$

$$\text{and } AD = 12\frac{1}{2}.$$

$$\text{Therefore } DF = \frac{10}{12} \cdot 12\frac{1}{2} \\ = 10\frac{5}{12}.$$

Quadratic equations solved in Heron.

We have already met with one such equation (in *Metrica* III. 4), namely $x^2 - 14x + 46\frac{2}{3} = 0$, the result only ($x = 8\frac{1}{2}$) being given. There are others in the *Geometrica* where the process of solution is shown.

(1) *Geometrica* 24, 3 (Heib.). 'Given a square such that the sum of its area and perimeter is 896 feet: to separate the area from the perimeter': i.e. $x^2 + 4x = 896$. Heron takes half of 4 and adds its square, completing the square on the left side.

(2) *Geometrica* 21, 9 and 24, 46 (Heib.) give one and the same equation, *Geom.* 24, 47 another like it. 'Given the sum of the diameter, perimeter and area of a circle, to find each of them.'

The two equations are

$$\frac{1}{4}d^2 + 2d = 212,$$

and

$$\frac{1}{4}d^2 + 2d = 67\frac{1}{2}.$$

Our usual method is to begin by dividing by $\frac{1}{4}$ throughout, so as to leave d^2 as the first term. Heron's is to multiply by such a number as will leave a square as the first term. In this case he multiplies by 164, giving $11^2d^2 + 58 \cdot 11d = 212 \cdot 164$ or $67\frac{1}{2} \cdot 164$ as the case may be. Completing the square, he obtains $(11d + 29)^2 = 32648 + 841$ or $10395 + 841$. Thus $11d + 29 = \sqrt{33489}$ or $\sqrt{11236}$, that is, 183 or 106. Thus $11d = 154$ or 77, and $d = 14$ or 7, as the case may be.

Indeterminate problems in the Geometrica.

Some very interesting indeterminate problems are now included by Heiberg in the *Geometrica*.¹ Two of them (chap. 24, 1-2) were included in the *Geöponicus* in Hultsch's edition (sections 78, 79); the rest are new, having been found in the Constantinople manuscript from which Schöno edited the *Metrica*. As, however, these problems, to whatever period they belong, are more akin to algebra than to mensuration, they will be more properly described in a later chapter on Algebra.

¹ *Heronis Alexandrini opera*, vol. iv, p. 414. 28 sq.

The Dioptra (περὶ διόπτρας).

This treatise begins with a careful description of the *dioptra*, an instrument which served with the ancients for the same purpose as a theodolite with us (chaps. 1-5). The problems with which the treatise goes on to deal are (a) problems of 'heights and distances', (b) engineering problems, (c) problems of mensuration, to which is added (p. 34) a description of a 'hodometer', or taxameter, consisting of an arrangement of toothed wheels and endless screws on the same axes working on the teeth of the next wheels respectively. The book ends with the problem (p. 37), 'With a given force to move a given weight by means of interacting toothed wheels', which really belongs to mechanics, and was apparently added, like some other problems (e.g. 31, 'to measure the outflow of, i.e. the volume of water issuing from, a spring'), in order to make the book more comprehensive. The essential problems dealt with are as follows. To determine the difference of level between two given points (6), to draw a straight line connecting two points the one of which is not visible from the other (7), to measure the least breadth of a river (9), the distance of inaccessible points (10), the height of an inaccessible point (11), to determine the difference between the heights of two accessible points and the position of the straight line joining them (13), the depth of a ditch (14); to bore a tunnel through a mountain going straight from one mouth to the other (15), to sink a shaft through a mountain perpendicularly to a canal lying underneath (16); given a subterranean canal of any depth, to find on the ground above a point from which a vertical shaft must be sunk in order to reach a given point in the canal (for the purpose e.g. of removing an obstruction) (18); to construct a harbour on the model of a given segment of a circle, given the ends (17), to construct a vault so that it may have a spherical surface modelled on a given segment of a sphere (19). The mensuration problems include the following: to measure an irregular area, which is done by inscribing a polygonal figure and then drawing perpendiculars to the sides at intervals to meet the contour (23), or by drawing one straight line across the area and erecting perpendiculars from

that to meet the contour on both sides (24); given that all the boundary stones of a certain area have disappeared except two or three, but that the plan of the area is forthcoming, to determine the position of the lost boundary stones (25). Chaps. 26-8 remind us of the *Metrica*: to divide a given area into given parts by straight lines drawn from one point (26); to measure a given area without entering it, whether because it is thickly covered with trees, obstructed by houses, or entry is forbidden! (27); chaps. 28-30 = *Metrica* III. 7, III. 1, and I. 7, the last of these three propositions being the proof of the 'formula of Heron' for the area of a triangle in terms of the sides. Chap. 35 shows how to find the distance between Rome and Alexandria along a great circle of the earth by means of the observation of the same eclipse at the two places, the *analemma* for Rome, and a concave hemisphere constructed for Alexandria to show the position of the sun at the time of the said eclipse. It is here mentioned that the estimate by Eratosthenes of the earth's circumference in his book *On the Measurement of the Earth* was the most accurate that had been made up to date.¹ Some hold that the chapter, like some others which have no particular connexion with the real subject of the *Dioptra* (e.g. chaps. 31, 34, 37-8) were probably inserted by a later editor, 'in order to make the treatise as complete as possible'.²

The *Mechanics*.

It is evident that the *Mechanics*, as preserved in the Arabic, is far from having kept its original form, especially in Book I. It begins with an account of the arrangement of toothed wheels designed to solve the problem of moving a given weight by a given force; this account is the same as that given at the end of the Greek text of the *Dioptra*, and it is clearly the same description as that which Pappus³ found in the work of Heron entitled *Βαρουλκός* ('weight-lifter') and himself reproduced with a ratio of force to weight altered from 5:1000 to 4:160 and with a ratio of 2:1 substituted for 5:1 in the diameters of successive wheels. It would appear that the chapter from the *Βαρουλκός* was inserted in place of

¹ Heron, vol. iii, p. 302. 13-17.

² *Ib.*, p. 302. 9.

³ Pappus, viii, p. 1060 sq.

first chapter or chapters of the real *Mechanics* which had been lost. The treatise would doubtless begin with generalities introductory to mechanics such as we find in the (much interpolated) beginning of Pappus, Book VIII. It must then apparently have dealt with the properties of circles, cylinders, spheres with reference to their importance in mechanics; in Book II. 21 Heron says that the circle is of all figures the most movable and most easily moved, the same thing applying also to the cylinder and sphere, and he adds in support of this a reference to a proof 'in the preceding Book'. This reference may be to I. 21, but at the end of that chapter he says that 'cylinders, even when heavy, if placed on the ground so that they touch it in one line only, are easily moved, and the same is true of spheres also, a matter which we have already discussed'; the discussion may have come earlier in the Book, in a chapter now lost.

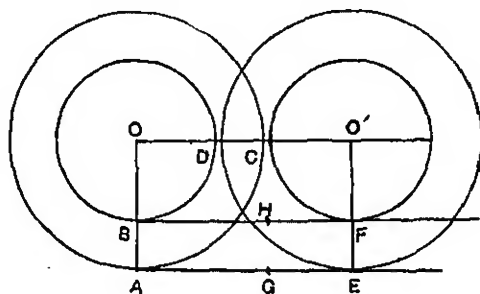
The treatise, beginning with chap. 2 after the passage interpolated from the *Βαρουλκός*, is curiously disconnected. Chaps. 2-7 discuss the motion of circles or wheels, equal or unequal, moving on different axes (e.g. interacting toothed wheels), or fixed on the same axis, much after the fashion of Aristotelian *Mechanical problems*.

Aristotle's Wheel.

In particular (chap. 7) Heron attempts to explain the puzzle of the 'Wheel of Aristotle', which remained a puzzle up to quite modern times, and gave rise to the proverb, 'rotam Aristotelis quis torquere, quo magis torqueretur'.¹ 'The question is', says the Aristotelian problem 24, 'why does the greater circle roll an equal distance with the lesser circle when they are placed about the same centre, whereas, when they roll separately, as the radius of one is to the size of the other, so are the straight lines reversed by them to one another?'² Let AC , BD be quadrants of circles with centre O bounded by the same radii, and draw tangents AE , BF at A and B . In the first case suppose the circle BD to roll along BF till D takes the position H ; then the radius ODC will be at right angles to AE , and C will be at G , a point such that AG is equal to BH . In the second

See Van Capello, *Aristotelis questiones mechanicae*, 1812, p. 268 sq.
 Arist. *Mechanica*, 855 a 28.

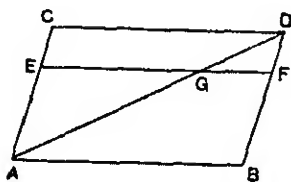
case suppose the circle AC to roll along AE till ODC takes the position $O'FE$; then D will be at F where $AE = BF$. And similarly if a whole revolution is performed and OBA is again perpendicular to AE . Contrary, therefore, to the principle that the greater circle moves quicker than the smaller on the same axis, it would appear that the movement of the



smaller in this case is as quick as that of the greater, since $BH = AG$, and $BF = AE$. Heron's explanation is that, e.g. in the case where the larger circle rolls on AE , the lesser circle maintains the same speed as the greater because it has *two* motions; for if we regard the smaller circle as merely fastened to the larger, and not rolling at all, its centre O will move to O' traversing a distance OO' equal to AE and BF ; hence the greater circle will take the lesser with it over an equal distance, the rolling of the lesser circle having no effect upon this.

The parallelogram of velocities.

Heron next proves the parallelogram of velocities (chap. 8); he takes the case of a rectangle, but the proof is applicable generally.



The way it is put is this. A point moves with uniform velocity along a straight line AB , from A to B , while at the same time AB moves with uniform velocity always parallel to itself with its extremity A describing the straight line AC .

Suppose that, when the point arrives at B , the straight line

reaches the position CD . Let EF be any intermediate position of AB , and G the position at the same instant of the moving point on it. Then clearly $AE:AC=EG:EF$; therefore $AE:EG=AC:EF=AC:CD$, and it follows that G lies on the diagonal AD , which is therefore the actual path of the moving point.

Chaps. 9-19 contain a digression on the construction of plane and solid figures similar to given figures but greater or less in a given ratio. Heron observes that the case of plane figures involves the finding of a mean proportional between two straight lines, and the case of solid figures the finding of two mean proportionals; in chap. 11 he gives his solution of the latter problem, which is preserved in Pappus and Eutocius as well, and has already been given above (vol. i, pp. 262-3). The end of chap. 19 contains, quite inconsequently, the construction of a toothed wheel to move on an endless screw, after which chap. 20 makes a fresh start with some observations on weights in equilibrium on a horizontal plane but tending to fall when the plane is inclined, and on the readiness of objects of cylindrical form which touch the plane on one line only.

Motion on an inclined plane.

When a weight is hanging freely by a rope over a pulley, the force applied to the other end of the rope less than the weight itself will keep it up, but, if the weight is placed on an inclined plane, and both the plane and the portion of the weight in contact with it are smooth, the case is different. Suppose, e.g., that a weight in the form of a cylinder is placed on an inclined plane so that the line in which they touch is horizontal; then the force required to be applied to a rope parallel to the line of greatest slope in the plane in order to keep the weight in equilibrium is less than the weight. For a vertical plane passing through the line of contact between the cylinder and the plane divides the cylinder into two equal parts, that on the downward side of the plane being the greater, so that the cylinder will tend to roll down; but the force required to support the cylinder is the 'equivalent', not of the weight of the whole cylinder, but of the difference

between the two portions into which the vertical plane cuts it (chap. 23).

On the centre of gravity.

This brings Heron to the centre of gravity (chap. 24). Here a definition by Posidonius, a Stoic, of the 'centre of gravity' or 'centre of inclination' is given, namely 'a point such that, if the body is hung up at it, the body is divided into two equal parts' (he should obviously have said 'divided by any vertical plane through the point of suspension into two equal parts'). But, Heron says, Archimedes distinguished between the 'centre of gravity' and the 'point of suspension', defining the latter as a point on the body such that, if the body is hung up at it, all the parts of the body remain in equilibrium and do not oscillate or incline in any direction. "Bodies", said Archimedes, "may rest (without inclining one way or another) with either a line, or only one point, in the body fixed". The 'centre of inclination', says Heron, 'is one single point in any particular body to which all the vertical lines through the points of suspension converge.' Comparing Simplicius's quotation of a definition by Archimedes in his *Κετροβαρικῆ*, to the effect that the centre of gravity is a certain point in the body such that, if the body is hung up by a string attached to that point, it will remain in its position without inclining in any direction,¹ we see that Heron directly used a certain treatise of Archimedes. So evidently did Pappus, who has a similar definition. Pappus also speaks of a body supported at a point by a vertical stick: if, he says, the body is in equilibrium, the line of the stick produced upwards must pass through the centre of gravity.² Similarly Heron says that the same principles apply when the body is supported as when it is suspended. Taking up next (chaps. 25-31) the question of 'supports', he considers cases of a heavy beam or a wall supported on a number of pillars, equidistant or not, even or not even in number, and projecting or not projecting beyond one or both of the extreme pillars, and finds how much of the weight is supported on each pillar. He says that Archimedes laid down the principles in his *Book on*

¹ Simplicius on *De caelo*, p. 548. 31-4, Heib.

² Pappus, viii, p. 1032. 5-24.

ports'. As, however, the principles are the same whether body is supported or hung up, it does not follow that this was a different work from that known as *περὶ ζυγῶν*. Chaps. 32-3, which are on the principles of the lever or of the wheel and axle, end with an explanation amounting to the fact that 'greater circles overpower smaller when their movement is about the same centre', a proposition which Pappus says Archimedes proved in his work *περὶ ζυγῶν*.¹ In chap. 32, Heron gives as his authority a proof given by Archimedes in the same work. With I. 33 may be compared II. 7, where Heron returns to the same subject of the greater and smaller circles moving about the same centre and states the proposition that weights reciprocally proportional to their radii are in equilibrium when suspended from opposite ends of the horizontal diameters, observing that Archimedes proved the proposition in his work 'On the equalization of inclination' (presumably *ἰσορροπίαι*).

Book II. The five mechanical powers.

Heron deals with the wheel and axle, the lever, the pulley, the wedge and the screw, and with combinations of these powers. The description of the powers comes first, chaps. 1-6, then, after II. 7, the proposition above referred to, and the theory of the several powers based upon it (chaps. 8-20). Applications to specific cases follow. Thus it is shown how to move a weight of 1000 talents by means of a force of 10 talents, first by the system of wheels described in the *Ποιητικὸς*, next by a system of pulleys, and thirdly by a combination of levers (chaps. 21-5). It is possible to combine different powers (other than the wedge) to produce the same result (chap. 29). The wedge and screw are discussed with reference to their angles (chaps. 30-1), and chap. 32 refers to the effect of friction.

Mechanics in daily life; queries and answers.

After a prefatory chapter (33), a number of queries resembling the Aristotelian problems are stated and answered (chap. 34), e.g. 'Why do waggons with two wheels carry weight more easily than those with four wheels?', 'Why

¹ Pappus, viii, p. 1068. 20-3.

do great weights fall to the ground in a shorter time than lighter ones?', 'Why does a stick break sooner when one puts one's knee against it in the middle?', 'Why do people use pincers rather than the hand to draw a tooth?', 'Why is it easy to move weights which are suspended?', and 'Why is it the more difficult to move such weights the farther the hand is away from them, right up to the point of suspension or a point near it?', 'Why are great ships turned by a rudder although it is so small?', 'Why do arrows penetrate armour or metal plates but fail to penetrate cloth spread out?'

Problems on the centre of gravity, &c.

II. 35, 36, 37 show how to find the centre of gravity of a triangle, a quadrilateral and a pentagon respectively. Then, assuming that a triangle of uniform thickness is supported by a prop at each angle, Heron finds what weight is supported by each prop, (a) when the props support the triangle only, (b) when they support the triangle plus a given weight placed at any point on it (chaps. 38, 39). Lastly, if known weights are put on the triangle at each angle, he finds the centre of gravity of the system (chap. 40); the problem is then extended to the case of any polygon (chap. 41).

Book III deals with the practical construction of engines for all sorts of purposes, machines employing pulleys with one, two, or more supports for lifting weights, oil-presses, &c.

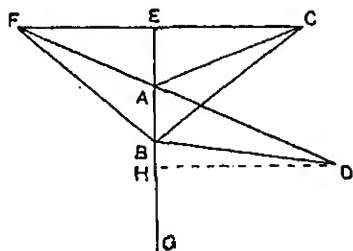
The Catoptrica.

This work need not detain us long. Several of the theoretical propositions which it contains are the same as propositions in the so-called *Catoptrica* of Euclid, which, as we have seen, was in all probability the work of Theon of Alexandria and therefore much later in date. In addition to theoretical propositions, it contains problems the purpose of which is to construct mirrors or combinations of mirrors of such shape as will reflect objects in a particular way, e.g. to make the right side appear as the right in the picture (instead of the reverse), to enable a person to see his back or to appear in the mirror head downwards, with face distorted, with three eyes or two noses, and so forth. Concave and convex

cylindrical mirrors play a part in these arrangements. The theory of course ultimately depends on the main propositions 4 and 5 that the angles of incidence and reflection are equal whether the mirror is plane or circular.

Archimedes' proof of equality of angles of incidence and reflection.

Let AB be a plane mirror, C the eye, D the object seen. The argument rests on the fact that nature 'does nothing in vain'. Thus light travels in a straight line, that is, by the shortest road. Therefore, even if the ray is a line broken at a point by reflection, it must be the shortest broken line of the kind connecting the eye with the object. Now, says Archimedes, I maintain that the shortest of the broken lines (taken at the mirror) which connect C and D is the line, as $CAED$, the parts of which make equal angles with the mirror. Produce DA and produce it to meet in F the perpendicular from C to AB . Let B be any point on the mirror other than A , join FB , BD .



Now $\angle EAF = \angle BAD$
 $= \angle CAE$, by hypothesis.

Therefore the triangles AEF , AEC , having two angles equal and AE common, are equal in all respects.

Therefore $CA = AF$, and $CA + AD = DF$.

Since $FE = EC$, and BE is perpendicular to FC , $BF = BC$.

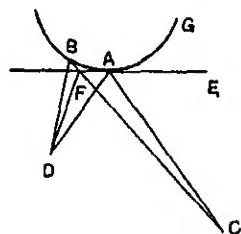
Therefore $CB + BD = FB + BD$
 $> FD$,
 i.e. $> CA + AD$.

The proposition was of course known to Archimedes. We learn from a scholium to the Pseudo-Euclidean *Catoptrica* that he proved it in a different way, namely by *reductio ad absurdum*, thus: Denote the angles CAE , DAB by α , β respectively. Then, α is $>$ or $<$ β . Suppose $\alpha > \beta$. Then,

reversing the ray so that the eye is at D instead of C , and the object at C instead of D , we must have $\beta > \alpha$. But β was less than α , which is impossible. (Similarly it can be proved that α is not less than β .) Therefore $\alpha = \beta$.

In the Pseudo-Euclidean *Catoptrica* the proposition is practically assumed; for the third assumption or postulate at the beginning states in effect that, in the above figure, if A be the point of incidence, $CE:EA = DH:HA$ (where DH is perpendicular to AB). It follows instantaneously (Prop. 1) that $\angle CAE = \angle DAH$.

If the mirror is the convex side of a circle, the same result follows *a fortiori*. Let CA, AD meet



the arc at equal angles, and CB, BD at unequal angles. Let AE be the tangent at A , and complete the figure. Then, says Heron, (the angles GAC, BAD being by hypothesis equal), if we subtract the equal angles GAE, BAF from the equal angles GAC, BAD (both pairs of angles being 'mixed', be it

observed), we have $\angle FAC = \angle FAD$. Therefore $CA + AD < CF + FD$ and *a fortiori* $< CB + BD$.

The problems solved (though the text is so corrupt in places that little can be made of it) were such as the following: 11, To construct a right-handed mirror (i.e. a mirror which makes the right side right and the left side left instead of the opposite); 12, to construct the mirror called *polytheonon* ('with many images'); 16, to construct a mirror inside the window of a house, so that you can see in it (while inside the room) everything that passes in the street; 18, to arrange mirrors in a given place so that a person who approaches cannot actually see either himself or any one else but can see any image desired (a 'ghost-seer').

XIX

PAPPUS OF ALEXANDRIA

We have seen that the Golden Age of Greek geometry ended with the time of Apollonius of Perga. But the influence of Euclid, Archimedes and Apollonius continued, and for some time there was a succession of quite competent mathematicians although not originating anything of capital importance, but up the tradition. Besides those who were known for particular investigations, e.g. of new curves or surfaces, there were such men as Geminus who, it cannot be doubted, were thoroughly familiar with the great classics. Geminus, as we have seen, wrote a comprehensive work of almost encyclopaedic character on the classification and content of mathematics, including the history of the development of each subject. At the beginning of the Christian era we see quite a different state of things. Except in sphaeric and astronomy (Menelaus and Ptolemy), production was limited to elementary textbooks of decidedly feeble quality. In the meantime it would seem that the study of higher geometry languished or was almost entirely in abeyance, until Pappus arose to revive interest in the subject. From the way in which he thinks it necessary to describe the contents of the classical works belonging to the *Treasury of Analysis*, for example, one would suppose that by his time many of them were, if not lost, completely forgotten, and that the great task which he set himself was the re-establishment of geometry on its former high plane of development. Presumably such interest as he was able to arouse soon flickered out, but for us his work has an invaluable value as constituting, after the works of the great mathematicians which have actually survived, the most important of all our sources.

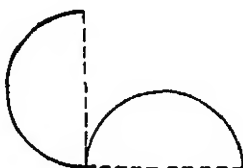
Date of Pappus.

Pappus lived at the end of the third century A.D. The authority for this date is a marginal note in a Leyden manuscript of chronological tables by Theon of Alexandria, where, opposite to the name of Diocletian, a scholium says, 'In his time Pappus wrote'. Diocletian reigned from 284 to 306, and this must therefore be the period of Pappus's literary activity. It is true that Suidas makes him a contemporary of Theon of Alexandria, adding that they both lived under Theodosius I (379-395). But Suidas was evidently not well acquainted with the works of Pappus; though he mentions a description of the earth by him and a commentary on four Books of Ptolemy's *Syntaxis*, he has no word about his greatest work, the *Synagoge*. As Theon also wrote a commentary on Ptolemy and incorporated a great deal of the commentary of Pappus, it is probable that Suidas had Theon's commentary before him and from the association of the two names wrongly inferred that they were contemporaries.

Works (commentaries) other than the *Collection*.

Besides the *Synagoge*, which is the main subject of this chapter, Pappus wrote several commentaries, now lost except for fragments which have survived in Greek or Arabic. One was a commentary on the *Elements* of Euclid. This must presumably have been pretty complete, for, while Proclus (on Eucl. I) quotes certain things from Pappus which may be assumed to have come in the notes on Book I, fragments of his commentary on Book X actually survive in the Arabic (see above, vol. i, pp. 154-5, 209), and again Eutocius in his note on Archimedes, *On the Sphere and Cylinder*, I. 13, says that Pappus explained in his commentary on the *Elements* how to inscribe in a circle a polygon similar to a polygon inscribed in another circle, which problem would no doubt be solved by Pappus, as it is by a scholiast, in a note on XII. 1. Some of the references by Proclus deserve passing mention. (1) Pappus said that the converse of Post. 4 (equality of all right angles) is not true, i.e. it is not true that all angles equal to a right angle are themselves right, since the 'angle' between the conterminous arcs of two semicircles which are equal and have their

ters at right angles and terminating at one point is to, but is not, a right angle.¹ (2) Pappus said that, in addition to the genuine axioms of Euclid, there were others accord about unequals added to equals and equals added to unequals. The axioms given by Pappus are (says Proclus) involved by the definitions, that 'all parts of the plane and of a straight line coincide with one another', that 'a point divides a line, a line a surface, and a surface a solid', and that 'the infinite is contained' in magnitudes both by addition and diminution'.² Pappus gave a pretty proof of Eucl. I. 5, which moderns have spoiled when introducing it into text-books. If AC are the equal sides in an isosceles triangle, Pappus compares the triangles ABC and ACB (i.e. as if he were comparing the triangle ABC seen from the front with the same triangle seen from the back), and shows that they satisfy the conditions of I. 4, so that they are equal in all respects, whence the result follows.³



Proclus at the end of his commentary on Euclid's *Data* refers to a commentary by Pappus on that book.

Pappus's commentary on Ptolemy's *Syntaxis* has already been mentioned (p. 274); it seems to have extended to six books, if not to the whole of Ptolemy's work. The *Fihrist* states that he also wrote a commentary on Ptolemy's *Planisphere*, which was translated into Arabic by Thābit b. Qurrah. Pappus himself alludes to his own commentary on the *Prolegomena* of Diodorus, in the course of which he used the method of Nicomedes for the purpose of trisecting an angle. We now come now to Pappus's great work.

The *Synagoge* or *Collection*.

(a) Character of the work; wide range.

Obviously written with the object of reviving the classical Greek geometry, it covers practically the whole field. It is,

¹ Proclus on Eucl. I, pp. 189-90.

² *Ib.*, pp. 197. 6-198. 15.

³ *Ib.*, pp. 249. 20-250. 12.

however, a handbook or guide to Greek geometry rather than an encyclopaedia; it was intended, that is, to be read with the original works (where still extant) rather than to enable them to be dispensed with. Thus in the case of the treatises included in the *Treasury of Analysis* there is a general introduction, followed by a general account of the contents, with lemmas, &c., designed to facilitate the reading of the treatises themselves. On the other hand, where the history of a subject is given, e.g. that of the problem of the duplication of the cube or the finding of the two mean proportionals, the various solutions themselves are reproduced, presumably because they were not easily accessible, but had to be collected from various sources. Even when it is some accessible classic which is being described, the opportunity is taken to give alternative methods, or to make improvements in proofs, extensions, and so on. Without pretending to great originality, the whole work shows, on the part of the author, a thorough grasp of all the subjects treated, independence of judgement, mastery of technique; the style is terse and clear; in short, Pappus stands out as an accomplished and versatile mathematician, a worthy representative of the classical Greek geometry.

(β) *List of authors mentioned.*

The immense range of the *Collection* can be gathered from a mere enumeration of the names of the various mathematicians quoted or referred to in the course of it. The greatest of them, Euclid, Archimedes and Apollonius, are of course continually cited, others are mentioned for some particular achievement, and in a few cases the mention of a name by Pappus is the whole of the information we possess about the person mentioned. In giving the list of the names occurring in the book, it will, I think, be convenient and may economize future references if I note in brackets the particular occasion of the reference to the writers who are mentioned for one achievement or as the authors of a particular book or investigation. The list in alphabetical order is: Apollonius of Perga, Archimedes, Aristaeus the elder (author of a treatise in five Books on the Elements of Conics or of 'five Books on Solid Loci connected with the conics'), Aristarchus of Samos (*On the*

On the distances of the sun and moon), Autolycus (*On the moving sphere*), Carpus of Antioch (who is quoted as having said that Archimedes wrote only one mechanical book, that of sphere-making, since he held the mechanical appliances which made him famous to be nevertheless unworthy of a written description: Carpus himself, who was known as *mechanicus*, applied geometry to other arts of this practical kind), Charmandrus (who added three simple and obvious loci to those which formed the beginning of the *Plane Loci* of Apollonius), Conon of Samos, the friend of Archimedes (cited as the propounder of a theorem about the spiral in a plane which Archimedes proved: this would, however, seem to be a mistake, as Archimedes says at the beginning of his treatise that he sent certain theorems, without proofs, to Conon, who could certainly have proved them had he lived), Demetrius of Alexandria (mentioned as the author of a work called 'Linear considerations', *γραμμικαὶ ἐπιστάσεις*, i.e. considerations on curves, as to which nothing more is known), Dinostratus, the brother of Menaechmus (cited, with Nicomedes, as having squared the curve of Hippias, to which they gave the name of *quadratrix*, *τετραγωνίστρουσα*, for the squaring of the circle), Eudorus (mentioned as the author of an *Analemma*), Eratosthenes (whose *mean-finder*, an appliance for finding two or any number of geometric means, is described, and who is further mentioned as the author of two Books 'On means' and of a work entitled 'Loci with reference to means'), Erycinus (from whose *Paradoxa* are quoted various problems seeming at first sight to be inconsistent with Eucl. I. 21, it being shown that straight lines can be drawn from two points on the base of a triangle to a point within the triangle which is together greater than the other two sides, provided that the points in the base may be points other than the extremities), Euclid, Geminus the mathematician (from whom is cited a remark on Archimedes contained in his book 'On the classification of the mathematical sciences', see above, p. 223), Heraclitus from whom Pappus quotes an elegant solution of a *νέσσις* (with reference to a square), Hermodorus (Pappus's son, to whom he dedicated Books VII, VIII of his *Collection*), Heron of Alexandria (whose mechanical works are extensively quoted in the *Collection*), Hierius the philosopher (a contemporary of Pappus,

who is mentioned as having asked Pappus's opinion on the attempted solution by 'plane' methods of the problem of the two means, which actually gives a method of approximating to a solution'), Hipparchus (quoted as practically adopting three of the hypotheses of Aristarchus of Samos), Megethion (to whom Pappus dedicated Book V of his *Collection*), Menelaus of Alexandria (quoted as the author of *Sphaerica* and as having applied the name *παράδοχος* to a certain curve), Nicomachus (on three means additional to the first three), Nicomedes, Pandrosion (to whom Book III of the *Collection* is dedicated), Pericles (editor of Euclid's *Data*), Philon of Byzantium (mentioned along with Heron), Philon of Tyana (mentioned as the discoverer of certain complicated curves derived from the interweaving of plectoid and other surfaces), Plato (with reference to the five regular solids), Ptolemy, Theodosius (author of the *Sphaerica* and *On Days and Nights*).

(γ) *Translations and editions.*

The first published edition of the *Collection* was the Latin translation by Commandinus (Venice 1589, but dated at the end 'Pisauri apud Hieronymum Concordiam 1588'; reissued with only the title-page changed 'Pisauri... 1602'). Up to 1876 portions only of the Greek text had appeared, namely Books VII, VIII in Greek and German, by C. J. Gerhardt, 1871, chaps. 33-105 of Book V, by Eisenmann, Paris 1824, chaps. 45-52 of Book IV in *Iosephi Torelli Veronensis Geometrica*, 1769, the remains of Book II, by John Wallis (in *Opera mathematica*, III, Oxford 1699); in addition, the restorers of works of Euclid and Apollonius from the indications furnished by Pappus give extracts from the Greek text relating to the particular works, Bréton le Champ on Euclid's *Porisms*, Halley in his edition of the *Conics* of Apollonius (1710) and in his translation from the Arabic and restoration respectively of the *De sectione rationis* and *De sectione spatii* of Apollonius (1706), Camerer on Apollonius's *Tactiones* (1795), Simson and Horsley in their restorations of Apollonius's *Plane Loci* and *Inclinationes* published in the years 1749 and 1770 respectively. In the years 1876-8 appeared the only com-

¹ See vol. i, pp. 268-70.

Greek text, with apparatus, Latin translation, commentary, appendices and indices, by Friedrich Hultsch; this edition is one of the first monuments of the revived interest in the history of Greek mathematics in the last half of the nineteenth century, and has properly formed the model for the definitive editions of the Greek text of the other works of the great Greek mathematicians, e.g. the editions of Euclid, Archimedes, Apollonius, &c., by Heiberg and others. The index in this edition of Pappus deserves special mention, as it largely serves as a dictionary of mathematical terms, used not only in Pappus but by the Greek mathematicians generally.

(8) *Summary of contents.*

At the beginning of the work, Book I and the first 13 propositions (out of 26) of Book II are missing. The first 13 propositions of Book II evidently, like the rest of the Book, follow Apollonius's method of working with very large numbers expressed in successive powers of the myriad, 10000. This system has already been described (vol. i, pp. 40, 54-7). The work of Apollonius seems to have contained 26 propositions (25 leading up to, and the 26th containing, the final continued multiplication).

Book III consists of four sections. Section (1) is a sort of survey of the problem of *finding two mean proportionals, in continued proportion, between two given straight lines.*

It begins with some general remarks about the distinction between theorems and problems. Pappus observes that, whereas the ancients called them all alike by one name, some regarding them all as problems and others as theorems, a clear distinction was drawn by those who favoured more exact terminology. According to the latter a problem is that in which it is proposed to *do* or *construct* something, a theorem is that in which, given certain hypotheses, we investigate that which follows from and is necessarily implied by them. Therefore he who propounds a theorem, no matter how he has become aware of the fact which is a necessary consequence of the premisses, must state, as the object of inquiry, the right result and no other. On the other hand, he who propounds

a problem may bid us do something which is in fact impossible, and that without necessarily laying himself open to blame or criticism. For it is part of the solver's duty to determine the conditions under which the problem is possible or impossible, and, 'if possible, when, how, and in how many ways it is possible'. When, however, a man professes to know mathematics and yet commits some elementary blunder, he cannot escape censure. Pappus gives, as an example, the case of an unnamed person 'who was thought to be a great geometer' but who showed ignorance in that he claimed to know how to solve the problem of the two mean proportionals by 'plane' methods (i.e. by using the straight line and circle only). He then reproduces the argument of the anonymous person, for the purpose of showing that it does not solve the problem as its author claims. We have seen (vol. i, pp. 269-70) how the method, though not actually solving the problem, does furnish a series of successive approximations to the real solution. Pappus adds a few simple lemmas assumed in the exposition.

Next comes the passage¹, already referred to, on the distinction drawn by the ancients between (1) *plane* problems or problems which can be solved by means of the straight line and circle, (2) *solid* problems, or those which require for their solution one or more conic sections, (3) *linear* problems, or those which necessitate recourse to higher curves still, curves with a more complicated and indeed a forced or unnatural origin (*βεβιασμένην*) such as spirals, quadratrices, cochloids and cissoids, which have many surprising properties of their own. The problem of the two mean proportionals, being a *solid* problem, required for its solution either conics or some equivalent, and, as conics could not be constructed by purely geometrical means, various mechanical devices were invented such as that of Eratosthenes (the *mean-finder*), those described in the *Mechanics* of Philon and Heron, and that of Nicomedes (who used the 'cochloidal' curve). Pappus proceeds to give the solutions of Eratosthenes, Nicomedes and Heron, and then adds a fourth which he claims as his own, but which is practically the same as that attributed by Eutocius to Sporus. All these solutions have been given above (vol. i, pp. 268-64, 266-8).

¹ Pappus, iii, p. 54. 7-22.

Section (2). *The theory of means.*

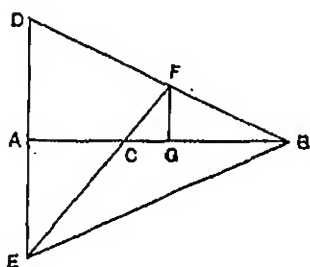
Next follows a section (pp. 69-105) on the theory of the different kinds of *means*. The discussion takes its origin in the statement of the 'second problem', which was that of exhibiting the three means' (i.e. the arithmetic, geometric and harmonic) 'in a semicircle'. Pappus first gives a construction by which another geometer (*ἄλλος τις*) claimed to have solved this problem, but he does not seem to have understood it, and returns to the same problem later (pp. 80-2).

In the meantime he begins with the definitions of the three means and then shows how, given any two of three numbers a, b, c in arithmetical, geometrical or harmonical progression, the third can be found. The definition of the mean of three terms a, b, c in harmonic progression being that it satisfies the relation $a:c = a-b:b-c$, Pappus gives alternative definitions for the arithmetic and geometric means in corresponding form, namely for the arithmetic mean $a:b = a-b:b-c$ and for the geometric $a:b = a-b:b-c$.

The construction for the harmonic mean is perhaps worth noting. Let AB, BG be two given straight lines. At A draw AE perpendicular to AB , and make DA, AE equal. Join BE . From G draw GF at right angles to AB meeting DB in F . Then EF meeting AB in C . Then AC is the required harmonic mean.

or

$$\begin{aligned} BG &= DA : FG \\ &= EA : FG \\ &= AC : CG \\ &= (AB - BC) : (BC - BG). \end{aligned}$$

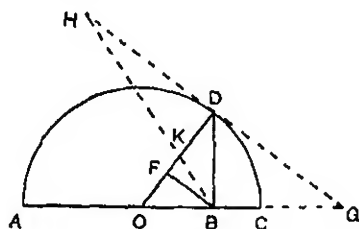


Similarly, by means of a like figure, we can find BG when AB, BC are given, and AB when BC, BG are given (in the latter case the perpendicular DE is drawn through G instead of A).

It then follows a proposition that, if the three means and the two extremes are represented in one set of lines, there must be five of them at least, and, after a set of five such lines have been found in the smallest possible integers, Pappus passes to

the problem of representing the three means with the respective extremes by *six* lines drawn in a semicircle.

Given a semicircle on the diameter AC , and B any point on the diameter, draw BD at right angles to AC . Let the tangent



at D meet AC produced in G , and measure DH along the tangent equal to DG . Join HB meeting the radius OD in K . Let BF be perpendicular to OD .

Then, exactly as above, it is shown that OK is a harmonic mean between OF and OD . Also BD is the geometric mean between AB , BC , while OC ($= OD$) is the arithmetic mean between AB , BC .

Therefore the six lines DO ($= OC$), OK , OF , AB , BC , BD supply the three means with the respective extremes.

But Pappus seems to have failed to observe that the 'certain other geometer', who has the same figure excluding the dotted lines, supplied the same in *five* lines. For he said that DF is 'a harmonic mean'. It is in fact the harmonic mean between AB , BC , as is easily seen thus.

Since ODB is a right-angled triangle, and BF perpendicular to OD ,

$$DF : BD = BD : DO,$$

or
$$DF \cdot DO = BD^2 = AB \cdot BC.$$

But
$$DO = \frac{1}{2}(AB + BC);$$

therefore
$$DF \cdot (AB + BC) = 2 AB \cdot BC.$$

Therefore
$$AB \cdot (DF - BC) = BC \cdot (AB - DF),$$

that is,
$$AB : BC = (AB - DF) : (DF - BC),$$

and DF is the harmonic mean between AB , BC .

Consequently the *five* lines DO ($= OC$), DF , AB , BC , BD exhibit all the three means with the extremes.

Pappus does not seem to have seen this, for he observes that the geometer in question, though saying that DF is a harmonic mean, does not say how it is a harmonic mean between what straight lines.

In the next chapters (pp. 84-104) Pappus, following Nicomachus and others, defines seven more means, three of which are ancient and the last four more modern, and shows how they can form all ten means as linear functions of α, β, γ , where β, γ are in geometrical progression. The exposition has already been described (vol. i, pp. 86-9).

Section (3). *The 'Paradoxes' of Erycinus.*

The third section of Book III (pp. 104-30) contains a series of propositions, all of the same sort, which are curious rather than geometrically important. They appear to have been taken direct from a collection of *Paradoxes* by one Erycinus.¹ The first set of these propositions (Props. 28-34) are connected with Eucl. I. 21, which says that, if from the extremities of the base of any triangle two straight lines be drawn meeting at any point within the triangle, the straight lines are together less than the two sides of the triangle other than the base, and contain a greater angle. It is pointed out that, if the straight lines are allowed to be drawn from points in the base other than the extremities, their sum may be greater than the other two sides of the triangle.

The first case taken is that of a right-angled triangle ABC right-angled at B . Draw AD to any point D on BC . Measure off DE equal to AB , bisect AE at F , and join FC . Then shall

$DF + FC$ be $> BA + AC$.

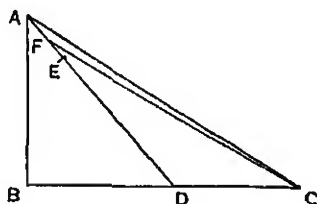
For $EF + FC = AF + FC > AC$.

Add DE and AB respectively, we have

$$DF + FC > BA + AC.$$

More elaborate propositions are next proved, such as the following.

In any triangle, except an equilateral triangle or an isosceles



¹ Pappus, iii, p. 106. 5-9.

triangle with base less than one of the other sides, it is possible to construct on the base and within the triangle two straight lines meeting at a point, the sum of which is *equal* to the sum of the other two sides of the triangle (Props. 29, 30).

2. In any triangle in which it is possible to construct two straight lines from the base to one internal point the sum of which is equal to the sum of the two sides of the triangle, it is also possible to construct two other such straight lines the sum of which is *greater* than that sum (Prop. 31).

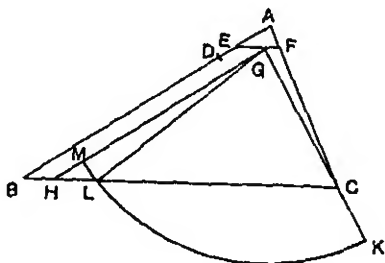
3. Under the same conditions, if the base is greater than either of the other two sides, two straight lines can be so constructed from the base to an internal point which are *respectively* greater than the other two sides of the triangle; and the lines may be constructed so as to be *respectively equal* to the two sides, if one of those two sides is less than the other and each of them is less than the base (Props. 32, 33).

4. The lines may be so constructed that their sum will bear to the sum of the two sides of the triangle any ratio less than 2 : 1 (Prop. 34).

As examples of the proofs, we will take the case of the scalene triangle, and prove the first and Part 1 of the third of the above propositions for such a triangle.

In the triangle ABC with base BC let AB be greater than AC .

Take D on BA such that $BD = \frac{1}{2}(BA + AC)$.



On DA between D and A take any point E , and draw EF parallel to BC . Let G be any point on EF ; draw GH parallel to AB and join GC .

the other, i.e. it is shown that broken lines, consisting of several straight lines, can be drawn with two points on the base of a triangle or parallelogram as extremities, and of greater total length than the remaining two sides of the triangle or three sides of the parallelogram.

Props. 40-2 show that triangles or parallelograms can be constructed with sides respectively greater than those of a given triangle or parallelogram but having a less area.

Section (4). *The inscribing of the five regular solids in a sphere.*

The fourth section of Book III (pp. 132-62) solves the problems of inscribing each of the five regular solids in a given sphere. After some preliminary lemmas (Props. 43-53), Pappus attacks the substantive problems (Props. 54-8), using the method of analysis followed by synthesis in the case of each solid.

(a) In order to inscribe a regular pyramid or tetrahedron in the sphere, he finds two circular sections equal and parallel to one another, each of which contains one of two opposite edges as its diameter. If d be the diameter of the sphere, the parallel circular sections have d' as diameter, where $d^2 = \frac{3}{2}d'^2$.

(b) In the case of the cube Pappus again finds two parallel circular sections with diameter d' such that $d^2 = \frac{3}{2}d'^2$; a square inscribed in one of these circles is one face of the cube and the square with sides parallel to those of the first square inscribed in the second circle is the opposite face.

(c) In the case of the octahedron the same two parallel circular sections with diameter d' such that $d^2 = \frac{3}{2}d'^2$ are used; an equilateral triangle inscribed in one circle is one face, and the opposite face is an equilateral triangle inscribed in the other circle but placed in exactly the opposite way.

(d) In the case of the icosahedron Pappus finds four parallel circular sections each passing through three of the vertices of the icosahedron; two of these are small circles circumscribing two opposite triangular faces respectively, and the other two circles are between these two circles, parallel to them, and equal to one another. The pairs of circles are determined in

way. If d be the diameter of the sphere, set out two right lines x, y such that d, x, y are in the ratio of the sides of the regular pentagon, hexagon and decagon respectively inscribed in one and the same circle. The smaller pair of lines have r as radius where $r^2 = \frac{1}{3}y^2$, and the larger pair have r' as radius where $r'^2 = \frac{1}{3}x^2$.

In the case of the dodecahedron the same four parallel circular sections are drawn as in the case of the icosahedron. The inscribed pentagons set the opposite way are inscribed in the smaller circles; these pentagons form opposite faces. The regular pentagons inscribed in the larger circles with vertices at the proper points (and again set the opposite way) determine the other five vertices of the inscribed dodecahedron.

The constructions are quite different from those in Euclid's *Elements* I. 13, 15, 14, 16, 17 respectively, where the problem is first to construct the particular regular solid and then to 'circumscribe it in a sphere', i.e. to determine the circumscribing sphere in each case. I have set out Pappus's propositions in full elsewhere.¹

Book IV.

At the beginning of Book IV the title and preface are missing, and the first section of the Book begins immediately with an enunciation. The first section (pp. 176-208) contains propositions 1-12 which, with the exception of Props. 8-10, seem to be isolated propositions given for their own sakes and not connected by any general plan.

Section (1). *Extension of the theorem of Pythagoras.*

The first proposition is of great interest, being the generalization of Eucl. I. 47, as Pappus himself calls it, which is by this time pretty widely known to mathematicians. The enunciation is as follows.

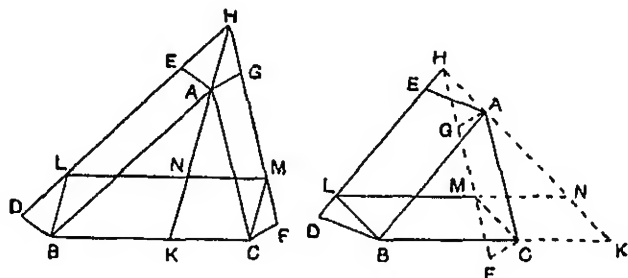
If ABC be a triangle and on AB, AC any parallelograms whatever be described, as $ABDE, ACFG$, and if DE, FG produced meet in H and HA be joined, then the parallelograms $ABDE, ACFG$ are together equal to the parallelogram

Vide notes to Euclid's propositions in The Thirteen Books of Euclid's Elements, pp. 478, 480, 477, 489-91, 501-3.

contained by BC , HA in an angle which is equal to the sum of the angles ABC , DHA .

Produce HA to meet BC in K , draw BL , CM parallel to KH meeting DE in L and FG in M , and join LMN .

Then $BLHA$ is a parallelogram, and HA is equal and parallel to BL .



Similarly HA , CM are equal and parallel; therefore BL , CM are equal and parallel.

Therefore $BLMC$ is a parallelogram; and its angle LBK is equal to the sum of the angles ABC , DHA .

Now $\square ABDE = \square BLHA$, in the same parallels,
 $= \square BLNK$, for the same reason.

Similarly $\square ACFG = \square ACMH = \square NKCM$.

Therefore, by addition, $\square ABDE + \square ACFG = \square BLMC$.

It has been observed (by Professor Cook Wilson¹) that the parallelograms on AB , AC need not necessarily be erected *outwards* from AB , AC . If one of them, e.g. that on AC , be drawn inwards, as in the second figure above, and Pappus's construction be made, we have a similar result with a negative sign, namely,

$$\begin{aligned}\square BLMC &= \square BLNK - \square CMNK \\ &= \square ABDE - \square ACFG.\end{aligned}$$

Again, if both $ABDE$ and $ACFG$ were drawn inwards, their sum would be equal to $BLMC$ drawn *outwards*. Generally, if the areas of the parallelograms described outwards are regarded as of opposite sign to those of parallelograms drawn inwards,

¹ *Mathematical Gazette*, vii, p. 107 (May 1913).

say that the algebraic sum of the three parallelograms is equal to zero.

Though Pappus only takes one case, as was the Greek habit, there is no reason to doubt that he was aware of the results in the other possible cases.

Props. 2, 3 are noteworthy in that they use the method and terminology of Eucl. X, proving that a certain line in one figure is the irrational called *minor* (see Eucl. X. 76), and that a certain line in another figure is 'the excess by which the square of a straight line exceeds the straight line which produces with a square area a medial whole' (Eucl. X. 77). The propositions 10-11 and 11-12 are quite interesting as geometrical exercises, though their bearing is not obvious: Props. 4 and 12 are remarkable in that they are cases of analysis followed by synthesis applied to the proof of *theorems*. Props. 8-10 belong to the class of *tangencies*, being the sort of propositions that would appear as particular cases in a book such as that of Apollonius on *contacts*; Prop. 8 shows that, if there are two equal circles and a given point outside both, the diameter of the circle passing through the point and touching both circles is bisected; the proof is in many places obscure and assumes results of the same kind as those given later à propos of Apollonius's treatise; Prop. 10 purports to show how, given two unequal circles touching one another two and two, to find the diameter of the circle including them and touching both.

Section (2). *On circles inscribed in the ἀρβηλος* (*'shoemaker's knife'*).

The next section (pp. 208-32), directed towards the demonstration of a theorem about the relative sizes of successive circles inscribed in the ἀρβηλος (shoemaker's knife), is extremely interesting and clever, and I wish that I had space to produce it completely. The ἀρβηλος, which we have already met with in Archimedes's 'Book of Lemmas', is defined thus. BC is the diameter of a semicircle BGC and is divided into two parts (in general unequal) at D ; two circles are described on BD , DC as diameters on the same side of BC as BGC is; the figure included between the three circles is the ἀρβηλος.

There is, says Pappus, on record an ancient proposition to the following effect. Let successive circles be inscribed in the $\alpha\rho\beta\eta\lambda\sigma\epsilon$ touching the semicircles and one another as shown in the figure on p. 376, their centres being $A, P, O \dots$. Then, if $p_1, p_2, p_3 \dots$ be the perpendiculars from the centres $A, P, O \dots$ on BC and $d_1, d_2, d_3 \dots$ the diameters of the corresponding circles,

$$p_1 = d_1, \quad p_2 = 2d_2, \quad p_3 = 3d_3 \dots$$

He begins by some lemmas, the course of which I shall reproduce as shortly as I can.

I. If (Fig. 1) two circles with centres A, C of which the former is the greater touch externally at B , and another circle with centre G touches the two circles at K, L respectively, then KL produced cuts the circle BL again in D and meets AC produced in a point E such that $AB:BC = AE:EC$. This is easily proved, because the circular segments DL, LK are similar, and CD is parallel to AG . Therefore

$$AB:BC = AK:CD = AE:EC.$$

Also

$$KE \cdot EL = EB^2.$$

$$\begin{aligned} \text{For } AE:EC &= AB:BC = AB:CF = (AE-AB):(EC-OF) \\ &= BE:EF. \end{aligned}$$

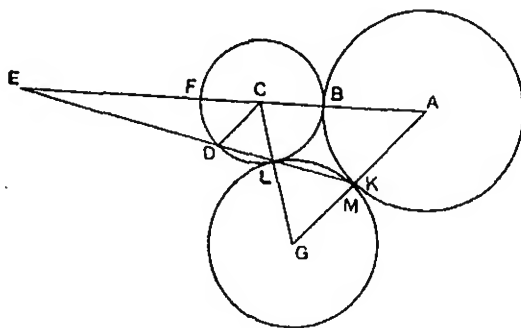


FIG 1.

But $AE:EC = KE:ED$; therefore $KE:ED = BE:EF$.

Therefore $KE \cdot EL:EL \cdot ED = BE^2:BE \cdot EF$.

And $EL \cdot ED = BE \cdot EF$; therefore $KE \cdot EL = EB^2$.

Let (Fig. 2) BC, BD , being in one straight line, be the diameters of two semicircles BGC, BED , and let any circle as F touch both semicircles, A being the centre of the circle. Let M be the foot of the perpendicular from A on BC , r the radius of the circle FGH . There are two cases according to whether A lies along BC or B lies between D and C ; i.e. in the first case the two semicircles are the outer and one of the inner circles of the $\alpha\rho\beta\eta\lambda\omicron\varsigma$, while in the second case they are two inner semicircles; in the latter case the circle FGH either includes the two semicircles or be entirely external to them. Now, says Pappus, it is to be proved that

$$(1) \quad BM : r = (BC + BD) : (BC - BD),$$

$$\text{case (2)} \quad BM : r = (BC - BD) : (BC + BD).$$

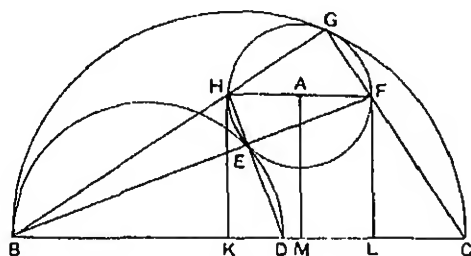


FIG. 2.

we will confine ourselves to the first case, represented in Figure (Fig. 2).

Draw through A the diameter HGF parallel to BC . Then, the circles BGC, HGF touch at G , and BC, HGF are parallel diameters, GHB, GFC are both straight lines.

E be the point of contact of the circles FGH and BED ; similarly, BEF, HED are straight lines.

BK, FL be drawn perpendicular to BC .

In the similar triangles BGC, BKH we have

$$BC : BG = BH : BK, \text{ or } CB \cdot BK = GB \cdot BH;$$

In the similar triangles BLF, BED

$$BF : BL = BD : BE, \text{ or } DB \cdot BL = FB \cdot BE.$$

But $GB.BH = FB.BE$;

therefore $CB.BK = DB.BI$,

or $BC:BD = BL:BK$.

Therefore $(BC + BD):(BC - BD) = (BL + BK):(BL - BK)$
 $= 2BM:KL$.

And $KL = HF = 2r$;

therefore $BM:r = (BC + BD):(BC - BD)$. (a)

It is next proved that $BK.LC = AM^2$.

For, by similar triangles BKH, FLC ,

$$BK:KH = FL:LC, \text{ or } BK.LC = KH.FL \\ = AM^2. \quad (b)$$

Lastly, since $BC:BD = BL:BK$, from above,

$$BC:CD = BL:KL, \text{ or } BL.CD = BC.KL \\ = BC.2r. \quad (c)$$

$$\text{Also } BD:CD = BK:KL, \text{ or } BK.CD = BD.KL \\ = BD.2r. \quad (d)$$

III. We now (Fig. 3) take any two circles touching the semicircles BGC, BED and one another. Let their centres be A and P , H their point of contact, d, d' their diameters respectively. Then, if AM, PN are drawn perpendicular to BC , Pappus proves that

$$(AM + d):d = PN:d'.$$

Draw BF' perpendicular to BC and therefore touching the semicircles BGC, BED at B . Join AP , and produce it to meet BF in F .

Now, by II. (a) above,

$$(BC + BD):(BC - BD) = BM:AH,$$

and for the same reason $= BN:PH$;

it follows that $AH:PH = BM:BN$
 $= FA:FP$.

Therefore (Lemma I), if the two circles touch the semi-circle BED in R, E respectively, FRE is a straight line and $FR = FH^2$.

4. $EF', FR = FB^2$; therefore $FH = FB$.

Now BH meets PN in O and MA produced in S , we have, similar triangles, $FH:FB=PH:PO=AH:AS$, whence $PO=FO$ and $SA=AH$, so that O, S are the intersections of PN, AM with the respective circles.

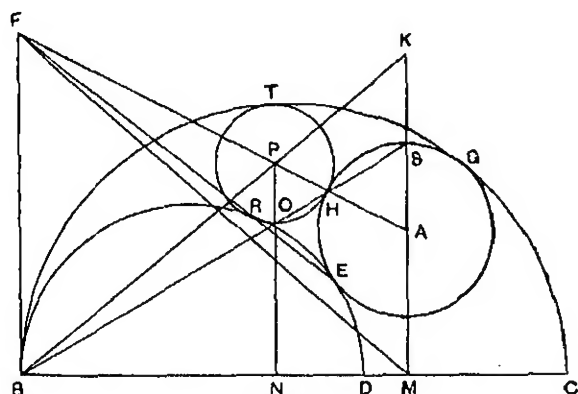


FIG. 3.

in BP , and produce it to meet MA in K .

$$\begin{aligned} BM : BN &= FA : FP \\ &= AH : PH, \text{ from above,} \\ &= AS : PO. \end{aligned}$$

$$BM:BN=BK:BP \\ =KS:PQ.$$

Therefore $KS = AS$, and $KA = d$, the diameter of the circle EHG .

$$MK:KS = PN:PO,$$

$$(AM + d) : \frac{1}{2}d = PN : \frac{1}{2}d',$$

$$(AM + d) : d = PN : d',$$

IV. We now come to the substantive theorem.

Let FGH be the circle touching all three semicircles (Fig. 4). We have then, as in Lemma II,

$$BC \cdot BK = BD \cdot BL,$$

and for the same reason (regarding FGH as touching the semicircles BGO , DUC)

$$BC \cdot CL = CD \cdot CK.$$

From the first relation we have

$$BC : BD = BL : BK,$$

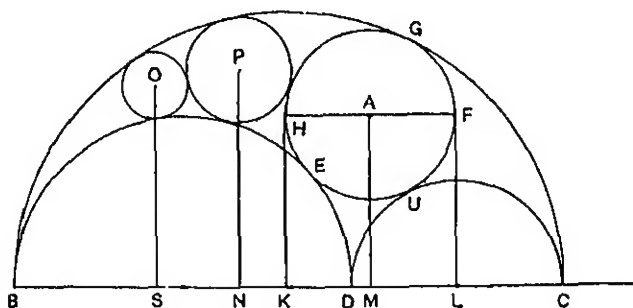


FIG. 4.

whence $DC : BD = KL : BK$, and inversely $BD : DC = BK : KL$, while, from the second relation, $BC : CD = CK : CL$,

whence $BD : DC = KL : CL$.

Consequently $BK : KL = KL : CL$,

or $BK \cdot LC = KL^2$.

But we saw in Lemma II (b) that $BK \cdot LC = AM^2$.

Therefore $KL = AM$, or $p_1 = d_1$.

For the second circle Lemma III gives us

$$(p_1 + d_1) : d_1 = p_2 : d_2,$$

whence, since $p_1 = d_1$, $p_2 = 2d_2$.

For the third circle

$$(p_2 + d_2) : d_2 = p_3 : d_3,$$

whence $p_3 = 3d_3$.

And so on *ad infinitum*.

The same proposition holds when the successive circles, instead of being placed between the large and one of the small semicircles, come down between the two small semicircles.

Pappus next deals with special cases (1) where the two smaller semicircles become straight lines perpendicular to the diameter of the other semicircle at its extremities, (2) where to replace one of the smaller semicircles by a straight line through D at right angles to BC , and lastly (3) where instead of the semicircle DUC we simply have the straight line DC and make the first circle touch it and the two other semicircles.

Pappus's propositions of course include as particular cases the partial propositions of the same kind included in the 'Book of Lemmas' attributed to Archimedes (Props. 5, 6); cf. p. 102.

Sections (3) and (4). *Methods of squaring the circle, and of trisecting (or dividing in any ratio) any given angle.*

The last sections of Book IV (pp. 234–302) are mainly devoted to the solutions of the problems (1) of squaring or rectifying the circle and (2) of trisecting any given angle or dividing it into two parts in any ratio. To this end Pappus gives a short account of certain curves which were used for the purpose.

(a) *The Archimedean spiral.*

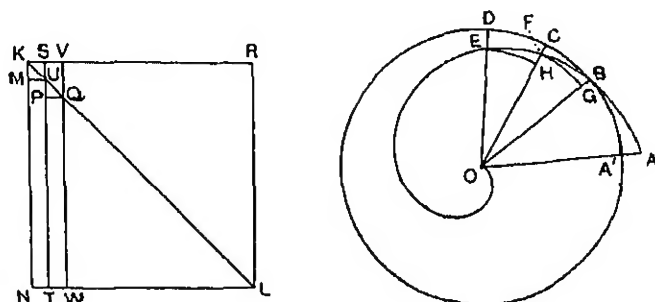
He begins with the spiral of Archimedes, proving some of the fundamental properties. His method of finding the area included (1) between the first turn and the initial line, (2) between any radius vector on the first turn and the curve, is worth giving because it differs from the method of Archimedes. It is the area of the whole first turn which Pappus works out in detail. We will take the area up to the radius vector OB , say.

With centre O and radius OB draw the circle $A'BCD$.

Let BC be a certain fraction, say $1/n$ th, of the arc $BCDA'$, and CD the same fraction, OC, OD meeting the spiral in F, E respectively. Let KS, SV be the same fraction of a straight line KR , the side of a square $KNLR$. Draw ST', VW parallel to KN meeting the diagonal KL of the square in U, Q respectively, and draw MU, PQ parallel to KR .

With O as centre and OE, OF as radii draw arcs of circles meeting OF, OB in H, G respectively.

For brevity we will now denote a cylinder in which r is the radius of the base and h the height by (cyl. r, h) and the cone with the same base and height by (cone r, h).



By the property of the spiral,

$$\begin{aligned} OB : BG &= (\text{arc } A'DCB) : (\text{arc } CB) \\ &= RK : KS \\ &= NK : KM, \end{aligned}$$

whence $OB : OG = NK : NM.$

Now

$$\begin{aligned} (\text{sector } OBC) : (\text{sector } OGF) &= OB^2 : OG^2 = NK^2 : NM^2 \\ &= (\text{cyl. } KN, NT) : (\text{cyl. } MN, NT). \end{aligned}$$

Similarly

$$(\text{sector } OCD) : (\text{sector } OEH) = (\text{cyl. } ST, TW) : (\text{cyl. } PT, TW),$$

and so on.

The sectors $OBC, OCD \dots$ form the sector $OA'DB$, and the sectors $OFG, OEH \dots$ form a figure inscribed to the spiral. In like manner the cylinders $(KN, TN), (ST, TW) \dots$ form the cylinder (KN, NL) , while the cylinders $(MN, NT), (PT, TW) \dots$ form a figure inscribed to the cone (KN, NL) .

Consequently

$$\begin{aligned} (\text{sector } OA'DB) : (\text{fig. inscr. in spiral}) \\ = (\text{cyl. } KN, NL) : (\text{fig. inscr. in cone } KN, NL). \end{aligned}$$

have a similar proportion connecting a figure circumscribed to the spiral and a figure circumscribed to the cone.

increasing n the inscribed and circumscribed figures can be pressed together, and by the usual method of exhaustion we have ultimately

$$\text{area } OA'DB : (\text{area of spiral}) = (\text{cyl. } KN, NL) : (\text{cone } KN, NL) \\ = 3:1,$$

area of spiral cut off by $OB = \frac{1}{3} \{\text{sector } OA'DB\}$.

The ratio of the sector $OA'DB$ to the complete circle is that of the angle which the radius vector describes in passing from position OA to the position OB to four right angles, that is, by the property of the spiral, $r:a$, where $r = OB$, $a = OA$.

$$\text{Therefore } (\text{area of spiral cut off by } OB) = \frac{1}{3} \frac{r}{a} \cdot \pi r^2.$$

Similarly the area of the spiral cut off by any other radius

$$\text{is } r' = \frac{1}{3} \frac{r'}{a} \cdot \pi r'^2.$$

Therefore (as Pappus proves in his next proposition) the area is to the second as r^3 to r'^3 .

Considering the areas cut off by the radii vectores at the angles π , $\frac{3}{2}\pi$ and 2π respectively, we see that the areas are in the ratio of $(\frac{1}{2})^3$, $(\frac{1}{2})^3$, $(\frac{3}{2})^3$, 1 or 1, 8, 27, 64, so that the areas of the spiral included in the four quadrants are in the ratio 7, 19, 37 (Prop. 22).

(β) *The conchoid of Nicomedes.*

The conchoid of Nicomedes is next described (chaps. 26-7). It is shown (chaps. 28, 29) how it can be used to find two geometric means between two straight lines, and consequently to find a cube having a given ratio to a given cube (see vol. i, pp. 260-2 and pp. 238-40, where I have also mentioned Pappus's remark that the conchoid which he describes is the conchoid, while there also exist a *second*, a *third* and a *fourth* which are of use for other theorems).

(γ) *The quadratrix.*

The quadratrix is taken next (chaps. 30-2), with Simplicius questioning the construction as involving a *petitio*

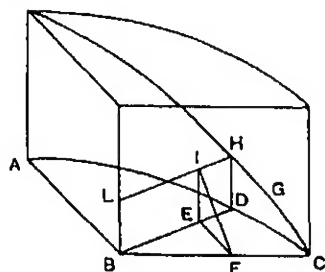
principii. Its use for squaring the circle is attributed to Dinostratus and Nicomedes. The whole substance of this subsection is given above (vol. i, pp. 226-30).

*Two constructions for the quadratrix by means of
'surface-loci'.*

In the next chapters (chaps. 33, 34, Props. 28, 29) Pappus gives two alternative ways of producing the *quadratrix* 'by means of surface-loci', for which he claims the merit that they are geometrical rather than 'too mechanical' as the traditional method (of Hippias) was.

(1) The first method uses a cylindrical helix thus.

Let ABC be a quadrant of a circle with centre B , and let BD be any radius. Suppose that EF , drawn from a point E on the radius BD perpendicular to BC , is (for all such radii) in a given ratio to the arc DC .



'I say', says Pappus, 'that the locus of E is a certain curve.'

Suppose a right cylinder erected from the quadrant and a cylindrical helix OGH drawn upon its surface. Let DH be the generator of this cylinder through D , meeting the helix in H . Draw BL , EI at right angles to the plane of the quadrant, and draw IHL parallel to BD .

Now, by the property of the helix, $EI (= DH)$ is to the arc CD in a given ratio. Also $EF : (\text{arc } CD) = \text{a given ratio}$.

Therefore the ratio $EF : EI$ is given. And since EF , EI are given in position, FI is given in position. But FI is perpendicular to BC . Therefore FI is in a plane given in position, and so therefore is I .

But I is also on a certain surface described by the line IHL , which moves always parallel to the plane ABC , with one extremity L on BL and the other extremity H on the helix. Therefore I lies on the intersection of this surface with the plane through FI .

Hence I lies on a certain curve. Therefore E , its projection on the plane ABC , also lies on a curve.

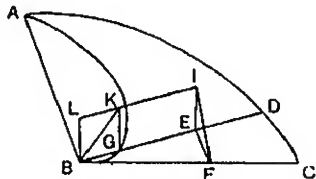
In the particular case where the given ratio of EF to the arc CD is equal to the ratio of BA to the arc CA , the locus of E is a *quadratrix*.

[The surface described by the straight line LH is a *plectoid*. The shape of it is perhaps best realized as a *continuous spiral staircase*, i.e. a spiral staircase with infinitely small steps. The *quadratrix* is thus produced as the orthogonal projection of the curve in which the plectoid is intersected by a plane through BC inclined at a given angle to the plane ABC . It is not difficult to verify the result analytically.]

(2) The second method uses a right cylinder the base of which is an Archimedean spiral.

Let ABC be a quadrant of a circle, as before, and EF , perpendicular at F to BC , a straight line of such length that EF is to the arc DC as AB is to the arc ADC .

Let a point on AB move uniformly from A to B while, in the same time, AB itself revolves uniformly about B from the position BA to the position BC . The point thus describes the spiral AGB . If the spiral cuts BD in G ,



$$BA : BG = (\text{arc } ADC) : (\text{arc } DC),$$

or $BG : (\text{arc } DC) = BA : (\text{arc } ADO).$

Therefore $BG = EF$.

Draw GK at right angles to the plane ABC and equal to BG . Then GK , and therefore K , lies on a right cylinder with the spiral as base.

But BK also lies on a conical surface with vertex B such that its generators all make an angle of $\frac{1}{4}\pi$ with the plane ABC .

Consequently K lies on the intersection of two surfaces, and therefore on a curve.

Through K draw LKI parallel to BD , and let BL , EI be at right angles to the plane ABC .

Then LKI , moving always parallel to the plane ABC , with one extremity on BL and passing through K on a certain

curve, describes a certain plectoid, which therefore contains the point I .

Also $IE = EF$, IF is perpendicular to BC , and hence IF , and therefore I , lies on a fixed plane through BC inclined to ABC at an angle of $\frac{1}{4}\pi$.

Therefore I , lying on the intersection of the plectoid and the said plane, lies on a certain curve. So therefore does the projection of I on ABC , i.e. the point E .

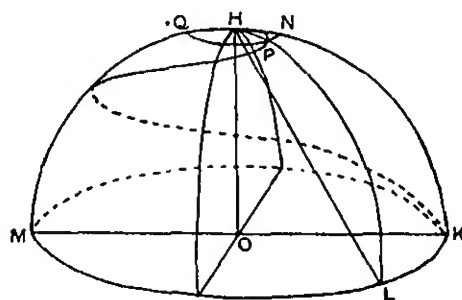
The locus of E is clearly the *quadratrix*.

[This result can also be verified analytically.]

(δ) *Digression: a spiral on a sphere.*

Prop. 30 (chap. 35) is a digression on the subject of a certain spiral described on a sphere, suggested by the discussion of a spiral in a plane.

Take a hemisphere bounded by the great circle KLM , with H as pole. Suppose that the quadrant of a great circle HNK revolves uniformly about the radius HO so that K describes the circle KLM and returns to its original position at K , and suppose that a point moves uniformly at the same



time from H to K at such speed that the point arrives at K at the same time that HNK resumes its original position. The point will thus describe a spiral on the surface of the sphere between the points H and K as shown in the figure.

Pappus then sets himself to prove that the portion of the surface of the sphere cut off towards the pole between the spiral and the arc HNK is to the surface of the hemisphere in

Now (sector HPN on sphere) : (sector HKL on sphere)

$$= (\text{chord } HP)^2 : (\text{chord } HL)^2$$

(a consequence of Archimedes, *On Sphere and Cylinder*, I. 42).

$$\begin{aligned}\text{And} \quad HP^2 : HL^2 &= CB^2 : CA^2 \\ &= CB^2 : CE^2.\end{aligned}$$

Therefore

$$(\text{sector } HPN) : (\text{sector } HKL) = (\text{sector } CBG) : (\text{sector } CEF).$$

Similarly, if the arc LL' be taken equal to the arc KL and the great circle through H, L' cuts the spiral in P' , and a small circle described about H and through P' meets the arc HPL in p ; and if likewise the arc BB' is made equal to the arc BC , and CB' is produced to meet AF' in E' , while again a circular arc with C as centre and CB' as radius meets CE in b ,

$$(\text{sector } HP'p \text{ on sphere}) : (\text{sector } HLL' \text{ on sphere})$$

$$= (\text{sector } CB'b) : (\text{sector } C'E'E).$$

And so on.

Ultimately then we shall get a figure consisting of sectors on the sphere circumscribed about the area S of the spiral and a figure consisting of sectors of circles circumscribed about the segment CBA ; and in like manner we shall have inscribed figures in each case similarly made up.

The method of exhaustion will then give

$$\begin{aligned}S : (\text{surface of hemisphere}) &= (\text{segmt. } ABC) : (\text{sector } CAF) \\ &= (\text{segmt. } ABC) : (\text{sector } DAO).\end{aligned}$$

[We may, as an illustration, give the analytical equivalent of this proposition. If ρ, ω be the spherical coordinates of P with reference to H as pole and the arc HNK as polar axis, the equation of Pappus's curve is obviously $\omega = 4\rho$.

If now the radius of the sphere is taken as unity, we have as the element of area

$$dA = d\omega (1 - \cos \rho) = 4 d\rho (1 - \cos \rho).$$

$$\text{Therefore} \quad A = \int_0^{1\pi} 4 d\rho (1 - \cos \rho) = 2\pi - 4.$$

Therefore

$$\frac{A}{(\text{surface of hemisphere})} = \frac{2\pi - 4}{2\pi} = \frac{\frac{1}{2}\pi - \frac{1}{2}}{\frac{1}{2}\pi} \\ = \frac{(\text{segment } ABC)}{(\text{sector } DABC)}.]$$

The second part of the last section of Book IV (chaps. 36-41, 70-302) is mainly concerned with the problem of trisecting any given angle or dividing it into parts in any given ratio.

Pappus begins with another account of the distinction between *plane*, *solid* and *linear* problems (cf. Book III, chaps. 1-10) according as they require for their solution (1) the straight line and circle only, (2) conics or their equivalent, or (3) higher curves still, 'which have a more complicated and less natural (or unnatural) origin, being produced from more complicated surfaces and involved motions. Such are the curves which are discovered in the so-called *loci on surfaces*, as well as others more complicated still and many in number considered by Demetrius of Alexandria in his *Linear constructions* and by Philon of Tyana by means of the intersection of plectoids and other surfaces of all sorts, all of which possess many remarkable properties peculiar to them. Some of these curves have been thought by the more recent geometers to be worthy of considerable discussion; one of them is the *Archimedean* curve, which also received from Menelaus the name of the *Archimedean* curve. Others of the same class are spirals, cuspids, cochloids and cissoids.' He adds the often-quoted remark on the error committed by geometers when they attempt to solve a problem by means of an 'inappropriate class' (of curves or its equivalent), illustrating this by the use in Pappus, Book V, of a rectangular hyperbola for finding the normals to a *parabola* passing through one point (where a circle would serve the purpose), and by the assumption by Archimedes of a *solid revolutis* in his book *On Spirals* (above, pp. 65-8).

Trisection (or division in any ratio) of any angle.

The method of trisecting any angle based on a certain *revolutis* next described, with the solution of the *revolutis* itself by

means of a hyperbola which has to be constructed from certain data, namely the asymptotes and a certain point through which the curve must pass (this easy construction is given in Prop. 33, chap. 41-2). Then the problem is directly solved (chaps. 43, 44) by means of a hyperbola in two ways practically equivalent, the hyperbola being determined in the one case by the ordinary Apollonian property, but in the other by means of the *focus-directrix* property. Solutions follow of the problem of dividing any angle in a given ratio by means (1) of the *quadratrix*, (2) of the spiral of Archimedes (chaps. 45, 46). All these solutions have been sufficiently described above (vol. i, pp. 235-7, 241-3, 225-7).

Some problems follow (chaps. 47-51) depending on these results, namely those of constructing an isosceles triangle in which either of the base angles has a given ratio to the vertical angle (Prop. 37), inscribing in a circle a regular polygon of any number of sides (Prop. 38), drawing a circle the circumference of which shall be equal to a given straight line (Prop. 39), constructing on a given straight line AB a segment of a circle such that the arc of the segment may have a given ratio to the base (Prop. 40), and constructing an angle incommensurable with a given angle (Prop. 41).

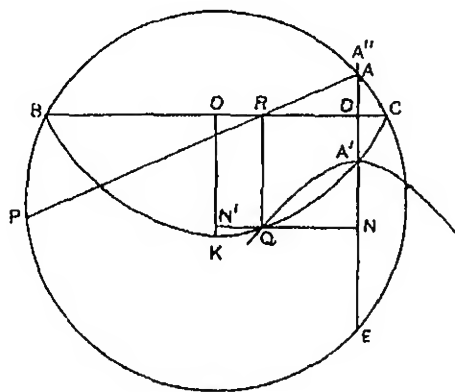
Section (5). *Solution of the νεῦσις of Archimedes, 'On Spirals', Prop. 8, by means of conics.*

Book IV concludes with the solution of the νεῦσις which, according to Pappus, Archimedes unnecessarily assumed in *On Spirals*, Prop. 8. Archimedes's assumption is this. Given a circle, a chord (BC) in it less than the diameter, and a point A on the circle the perpendicular from which to BC cuts BC in a point D such that $BD > DC$ and meets the circle again in E , it is possible to draw through A a straight line ARP cutting BC in R and the circle in P in such a way that RP shall be equal to DE (or, in the phraseology of νεῦσις, to place between the straight line BC and the circumference of the circle a straight line equal to DE and *verging* towards A).

Pappus makes the problem rather more general by not requiring PR to be equal to DE , but making it of any given

(consistent with a real solution). The problem is best solved by means of analytical geometry.

$BD = a$, $DC = b$, $AD = c$ (so that $DE = ab/c$), we have



Find the point R on BC such that AR produced solves the problem by making PR equal to k , say.

$DR = x$. Then, since $BR, RC = PR, RA$, we have

$$(a-x)(b+x) = k\sqrt{c^2+x^2}.$$

obvious expedient is to put y for $\sqrt{c^2 + x^2}$, when

$$(a-x)(b+x) = kx, \quad (1)$$

$$y^2 = c^2 + x^2. \quad (2)$$

se equations represent a parabola and a hyperbola
tively, and Pappus does in fact solve the problem by
of the intersection of a parabola and a hyperbola; one
preliminary lemmas is, however, again a little more
d. In the above figure y is represented by RQ .

first lemma of Pappus (Prop. 42, p. 298) states that, if a given point A any straight line be drawn meeting right line BC given in position in R , and if RQ be drawn at right angles to BC and of length bearing a given ratio to AR , the locus of Q is a *hyperbola*.

draw AD perpendicular to BC and produce it to A'

$QR:RA = A'D:DA =$ the given ratio.

Measure DA'' along DA equal to DA' .

Then, if QN be perpendicular to AD ,

$$(AR^2 - AD^2) : (QR^2 - A'D^2) = (\text{const.}),$$

that is, $QN^2 : A'N \cdot A''N = (\text{const.}),$

and the locus of Q is a hyperbola.

The equation of the hyperbola is clearly

$$x^2 = \mu(y^2 - c^2),$$

where μ is a constant. In the particular case taken by Archimedes $QR = RA$, or $\mu = 1$, and the hyperbola becomes the rectangular hyperbola (2) above.

The second lemma (Prop. 43, p. 300) proves that, if BC is given in length, and Q is such a point that, when QR is drawn perpendicular to BC , $BR \cdot RC = k \cdot QR$, where k is a given length, the locus of Q is a parabola.

Let O be the middle point of BC , and let OK be drawn at right angles to BC and of length such that

$$OC^2 = k \cdot KO.$$

Let QN' be drawn perpendicular to OK .

Then $QN'^2 = OR^2$

$$= OC^2 - BR \cdot RC$$

$$= k \cdot (KO - QR), \text{ by hypothesis,}$$

$$= k \cdot KN'.$$

Therefore the locus of Q is a parabola.

The equation of the parabola referred to DB , DE as axes of x and y is obviously

$$\left\{ \frac{1}{2}(a-b) - x \right\}^2 = k \cdot \left\{ \frac{(a+b)^2}{4k} - y \right\},$$

which easily reduces to

$$(a-x)(b+x) = ky, \text{ as above (1).}$$

In Archimedes's particular case $k = ab/c$.

To solve the problem then we have only to draw the parabola and hyperbola in question, and their intersection then gives Q , whence R , and therefore ARP , is determined.

Book V. Preface on the Sagacity of Bees.

characteristic of the great Greek mathematicians that, ever they were free from the restraint of the technical jargon of mathematics, as when for instance they had occasion to write a preface, they were able to write in language of the highest literary quality, comparable with that of the philosophers, historians, and poets. We have only to recall the introductions to Archimedes's treatises and the prefaces to the different Books of Apollonius's *Conics*. Heron, though very practical, is no exception when he has any general occasion, historical or other, to give. We have now to consider the like case in Pappus, namely the preface to Book V of the *Collection*. The editor, Hultsch, draws attention to the elegance and purity of the language and the careful writing; the style is illustrated by the studied avoidance of hiatus.¹ The subject is one which a writer of taste and imagination could not but naturally find attractive, namely the practical intelligence shown by bees in selecting the hexagonal form for the cells of the honeycomb. Pappus does not disappoint us; the preface is as attractive as the subject, and deserves to be translated.

It is of course to men that God has given the best and most perfect notion of wisdom in general and of mathematical science in particular, but a partial share in these things he has also granted to some of the unreasoning animals as well. To men, being endowed with reason, he vouchsafed that they should be able to do anything in the light of reason and demonstration, but to the other animals, while denying them reason, he granted that each of them should, by virtue of a certain natural instinct, obtain just so much as is needful to support life. This instinct may be observed to exist in very many other kinds of living creatures, but most of all in bees. In the first place their orderliness and their submission to the queens who direct their state are truly admirable, but much more admirable is their emulation, the cleanliness they observe in the making of honey, and the forethought and housewifely care they devote to its custody. Presumably because they know themselves to be entrusted with the task of bringing from the bees to the accomplished portion of mankind a share of

¹ Pappus, vol. iii, p. 1283.

ambrosia in this form, they do not think it proper to pour it carelessly on ground or wood or any other ugly and irregular material; but, first collecting the sweets of the most beautiful flowers which grow on the earth, they make from them, for the reception of the honey, the vessels which we call honeycombs, (with cells) all equal, similar and contiguous to one another, and hexagonal in form. And that they have contrived this by virtue of a certain geometrical forethought we may infer in this way. They would necessarily think that the figures must be such as to be contiguous to one another, that is to say, to have their sides common, in order that no foreign matter could enter the interstices between them and so defile the purity of their produce. Now only three rectilinear figures would satisfy the condition, I mean regular figures which are equilateral and equiangular; for the bees would have none of the figures which are not uniform. . . . There being then three figures capable by themselves of exactly filling up the space about the same point, the bees by reason of their instinctive wisdom chose for the construction of the honeycomb the figure which has the most angles, because they conceived that it would contain more honey than either of the two others.

'Bees, then, know just this fact which is of service to themselves, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material used in constructing the different figures. We, however, claiming as we do a greater share in wisdom than bees, will investigate a problem of still wider extent, namely that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest plane figure of all those which have a perimeter equal to that of the polygons is the circle.'

Book V then is devoted to what we may call *isoperimetry*, including in the term not only the comparison of the areas of different plane figures with the same perimeter, but that of the contents of different solid figures with equal surfaces.

Section (1). *Isoperimetry after Zenodorus.*

The first section of the Book relating to plane figures (chaps. 1-10, pp. 308-34) evidently followed very closely the exposition of Zenodorus *περὶ ἰσομέτρων σχημάτων* (see pp. 207-13, above); but before passing to solid figures Pappus inserts the proposition that *of all circular segments having*

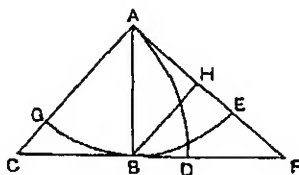
the same circumference the semicircle is the greatest, with some preliminary lemmas which deserve notice (chaps. 15, 16).

(1) ABC is a triangle right-angled at B . With C as centre and radius CA describe the arc AD cutting CB produced in D .

To prove that (R denoting a right angle)

$$(\text{sector } CAD) : (\text{area } ABD)$$

$$> R : \angle BCA.$$



Draw AF at right angles to CA meeting CD produced in F , and draw BH perpendicular to AF . With A as centre and AB as radius describe the arc GBE .

Now $(\text{area } EBF) : (\text{area } EBH) > (\text{area } EBF) : (\text{sector } ABE)$,

and, *componendo*, $\triangle FBH : (EBH) > \triangle ABF : (ABE)$.

But (by an easy lemma which has just preceded)

$$\triangle FBH : (EBH) = \triangle ABF : (ABD),$$

hence $\triangle ABF : (ABD) > \triangle ABF : (ABE)$,

and $(ABE) > (ABD)$.

Therefore $(ABE) : (ABG) > (ABD) : (ABG)$

$$> (ABD) : \triangle ABC, \text{ a fortiori.}$$

Therefore $\angle BAF : \angle BAC > (ABD) : \triangle ABC$,

hence, inversely, $\triangle ABC : (ABD) > \angle BAC : \angle BAF$.

and, *componendo*, $(\text{sector } ACD) : (ABD) > R : \angle BCA$.

[If α be the circular measure of $\angle BCA$, this gives (if $AC = b$)

$$\frac{1}{2} \alpha b^2 : (\frac{1}{2} \alpha b^2 - \frac{1}{2} \sin \alpha \cos \alpha \cdot b^2) > \frac{1}{2} \pi : \alpha,$$

$$2\alpha : (2\alpha - \sin 2\alpha) > \pi : 2\alpha;$$

that is, $\theta / (\theta - \sin \theta) > \pi / \theta$, where $0 < \theta < \pi$.]

(2) ABC is again a triangle right-angled at B . With C as centre and CA as radius draw a circle AD meeting BC produced in D . To prove that

$$(\text{sector } CAD) : (\text{area } ABD) > R : \angle ACD.$$

Draw AE at right angles to AC . With A as centre and AC as radius describe the circle FCE meeting AB produced in F and AE in E .

Then, since $\angle ACD > \angle CAE$, (sector ACD) $>$ (sector ACE).

Therefore $(ACD) : \triangle ABC > (ACE) : \triangle ABC$

$> (\triangle CFE) : (\triangle CFE)$, *a fortiori*,

$> \angle EAC : \angle CAB$.

Inversely,

$\triangle ABC : (ACD) < \angle CAB : \angle EAC$,

and, *componendo*,

$(ABD) : (ACD) < \angle EAB : \angle EAC$.

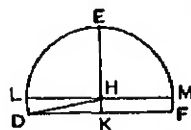
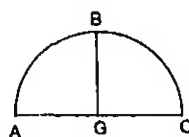
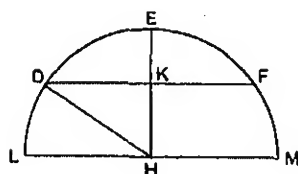
Inversely, $(ACD) : (ABD) > \angle EAC : \angle EAB$

$> R : \angle ACD$.

We come now to the application of these lemmas to the proposition comparing the area of a semicircle with that of other segments of equal circumference (chaps. 17, 18).

A semicircle is the greatest of all segments of circles which have the same circumference.

Let ABC be a semicircle with centre G , and DEF another segment of a circle such that the circumference DEF is equal



to the circumference ABC . I say that the area of ABC is greater than the area of DEF .

Let H be the centre of the circle DEF . Draw EHK , BG at right angles to DF , AC respectively. Join DH , and draw LHM parallel to DF .

$$\begin{aligned}
 LH:AG &= (\text{arc } LE):(\text{arc } AB) \\
 &= (\text{arc } LE):(\text{arc } DE) \\
 &= (\text{sector } LHE):(\text{sector } DHE).
 \end{aligned}$$

$$LH^2:AG^2 = (\text{sector } LHE):(\text{sector } AGB).$$

Therefore the sector LHE is to the sector AGB in the duplicate of that which the sector LHE has to the sector DHE .

Therefore

$$(\text{sector } LHE):(\text{sector } DHE) = (\text{sector } DHE):(\text{sector } AGB).$$

Now (1) in the case of the segment less than a semicircle

(2) in the case of the segment greater than a semicircle

$$(\text{sector } EDH):(\text{sector } EDK) > R:\angle DHE,$$

the lemmas (1) and (2) respectively.

That is,

$$(\text{sector } EDH):(\text{sector } EDK) > \angle LHE:\angle DHE$$

$$> (\text{sector } LHE):(\text{sector } DHE)$$

$$> (\text{sector } EDH):(\text{sector } AGB),$$

from above.

Therefore the half segment EDK is less than the half semicircle AGB , whence the semicircle ABC is greater than segment DEF .

We have already described the content of Zenodorus's *Prolegomena* (pp. 207-13, above) to which, so far as plane figures concerned, Pappus added nothing except the above proposition relating to segments of circles.

(2). *Comparison of volumes of solids having their surfaces equal. Case of the sphere.*

The portion of Book V dealing with solid figures begins § 20 with the statement that the philosophers who considered that the creator gave the universe the form of a sphere because that was the most beautiful of all shapes also asserted that the sphere is the greatest of all solid figures

which have their surfaces equal; this, however, they had not proved, nor could it be proved without a long investigation. Pappus himself does not attempt to prove that the sphere is greater than *all* solids with the same surface, but only that the sphere is greater than any of the five regular solids having the same surface (chap. 19) and also greater than either a cone or a cylinder of equal surface (chap. 20).

Section (3). *Digression on the semi-regular solids of Archimedes.*

He begins (chap. 19) with an account of the thirteen semi-regular solids discovered by Archimedes, which are contained by polygons all equilateral and all equiangular but not all similar (see pp. 98-101, above), and he shows how to determine the number of solid angles and the number of edges which they have respectively; he then gives them the go-by for his present purpose because they are not completely regular; still less does he compare the sphere with any irregular solid having an equal surface.

The sphere is greater than any of the regular solids which has its surface equal to that of the sphere.

The proof that the sphere is greater than any of the regular solids with surface equal to that of the sphere is the same as that given by Zenodorus. Let P be any one of the regular solids, S the sphere with surface equal to that of P . To prove that $S > P$. Inscribe in the solid a sphere s , and suppose that r is its radius. Then the surface of P is greater than the surface of s , and accordingly, if R is the radius of S , $R > r$. But the volume of S is equal to the cone with base equal to the surface of S , and therefore of P , and height equal to R ; and the volume of P is equal to the cone with base equal to the surface of P and height equal to r . Therefore, since $R > r$, volume of $S >$ volume of P .

Section (4). *Propositions on the lines of Archimedes, 'On the Sphere and Cylinder'.*

For the fact that the volume of a sphere is equal to the cone with base equal to the surface, and height equal to the radius,



the sphere, Pappus quotes Archimedes, *On the Sphere and Cylinder*, but thinks proper to add a series of propositions (Props. 20-43, pp. 362-410) on much the same lines as those of Archimedes and leading to the same results as Archimedes' for the surface of a segment of a sphere and of the whole sphere (Prop. 28), and for the volume of a sphere (Prop. 35). Prop. 36 (chap. 42) shows how to divide a sphere into two parts such that their surfaces are in a given ratio and Prop. 37 (chap. 43) proves that the volume as well as the surface of the cylinder circumscribing a sphere is $1\frac{1}{2}$ times of the sphere itself.

Among the lemmatic propositions in this section of the Book Props. 21, 22 may be mentioned. Prop. 21 proves that, if C, E be two points on the tangent at H to a semicircle such that $CH = HE$, and if CD, EF be drawn perpendicular to the diameter AB , then $(CD + EF) CE = AB \cdot DF$; Prop. 22 proves the same result where C, E are points on the semicircle, CD, EF are drawn perpendicular to AB , and EH is the chord of the semicircle subtending the arc which with CE makes up a semicircle; in this case $(CD + EF) CE = EH \cdot DF$. Both results may easily be seen to be the equivalent of the trigonometrical formula

$$\sin(x+y) + \sin(x-y) = 2 \sin x \cos y,$$

if certain different angles be taken as x, y ,

$$\frac{\sin x + \sin y}{\cos y - \cos x} = \cot \frac{1}{2}(x-y).$$

Proposition (5). *Of regular solids with surfaces equal, that is greater which has more faces.*

Returning to the main problem of the Book, Pappus shows that of the five regular solid figures assumed to have their surfaces equal, that is greater which has the more faces, so that the pyramid, the cube, the octahedron, the dodecahedron and the icosahedron of equal surface are, as regards solid content, in ascending order of magnitude (Props. 38-56). Pappus indicates (p. 410. 27) that 'some of the ancients' had omitted the proofs of these propositions by the analytical method; for himself, he will give a method of his own by

synthetical deduction, for which he claims that it is clearer and shorter. We have first propositions (with auxiliary lemmas) about the perpendiculars from the centre of the circumscribing sphere to a face of (*a*) the octahedron, (*b*) the icosahedron (Props. 39, 43), then the proposition that, if a dodecahedron and an icosahedron be inscribed in the same sphere, the same small circle in the sphere circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron (Prop. 48); this last is the proposition proved by Hypsicles in the so-called 'Book XIV of Euclid', Prop. 2, and Pappus gives two methods of proof, the second of which (chap. 56) corresponds to that of Hypsicles. Prop. 49 proves that twelve of the regular pentagons inscribed in a circle are together greater than twenty of the equilateral triangles inscribed in the same circle. The final propositions proving that the cube is greater than the pyramid with the same surface, the octahedron greater than the cube, and so on, are Props. 52-6 (chaps. 60-4). Of Pappus's auxiliary propositions, Prop. 41 is practically contained in Hypsicles's Prop. 1, and Prop. 44 in Hypsicles's last lemma; but otherwise the exposition is different.

Book VI.

On the contents of Book VI we can be brief. It is mainly astronomical, dealing with the treatises included in the so-called *Little Astronomy*, that is, the smaller astronomical treatises which were studied as an introduction to the great *Syntaxis* of Ptolemy. The preface says that many of those who taught the *Treasury of Astronomy*, through a careless understanding of the propositions, added some things as being necessary and omitted others as unnecessary. Pappus mentions at this point an incorrect addition to Theodosius, *Sphaerica*, III. 6, an omission from Euclid's *Phaenomena*, Prop. 2, an inaccurate representation of Theodosius, *On Days and Nights*, Prop. 4, and the omission later of certain other things as being unnecessary. His object is to put these mistakes right. Allusions are also found in the Book to Menelaus's *Sphaerica*, e.g. the statement (p. 476. 16) that Menelaus in his *Sphaerica* called a spherical triangle *τρίπλευρον*, *three-side*.

Sphaerica of Theodosius is dealt with at some length (chs. 1-26, Props. 1-27), and so are the theorems of Ptolemy *On the moving Sphere* (chaps. 27-9), Theodosius *Days and Nights* (chaps. 30-6, Props. 29-38), Aristarchus *On the sizes and distances of the Sun and Moon* (chaps. 37-40, Prop. 39 with two lemmas, which is not at the end and is not really proved), Euclid's *Optics* (chs. 41-52, Props. 42-54), and Euclid's *Phaenomena* (chaps. 55-61).

Problem arising out of Euclid's 'Optics'.

There is little in the Book of general mathematical interest but the following propositions which occur in the section on Euclid's *Optics*.

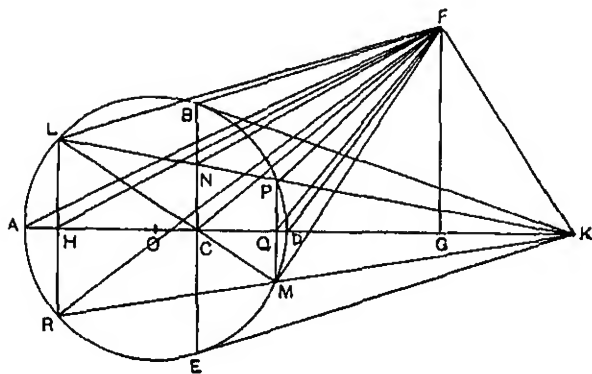
Two propositions are fundamental in solid geometry, namely:

1. If from a point A above a plane AB be drawn perpendicular to the plane, and if from B a straight line BD be drawn perpendicular to any straight line EF in the plane, then AD also be perpendicular to EF (Prop. 43).

2. If from a point A above a plane AB be drawn perpendicular to the plane not at right angles to it, and AM be drawn perpendicular to the plane (i.e. if BM be the orthogonal projection of BA on the plane), the angle ABM is the least of all the angles which AB makes with any straight lines through B , as BP , in the plane; the angle ABP increases as BP moves away from BM on either side; and, given any straight line BP making a certain angle with BA , only one other straight line in the plane will make the same angle with BA , namely a straight line BP' on the other side of BM making the same angle with BA as BP does (Prop. 44).

These are the first of a series of lemmas leading up to the main problem, the investigation of the apparent form of a circle as seen from a point outside its plane. In Prop. 50 (Euclid, *Optics*, 34) Pappus proves the fact that all the diameters of the circle will appear equal if the straight line drawn from the point representing the eye to the centre of the circle is either (a) at right angles to the plane of the circle, or if not at right angles to the plane of the circle, is equal

in length to the radius of the circle. In all other cases (Prop. 51 = Eucl. *Optics*, 35) the diameters will appear unequal. Pappus's other propositions carry further Euclid's remark that the circle seen under these conditions will appear deformed or distorted (*παρεσπασμένος*), proving (Prop. 53, pp. 588-92) that the apparent form will be an ellipse with its centre not, 'as some think', at the centre of the circle but at another point in it, determined in this way. Given a circle $ABDE$ with centre O , let the eye be at a point F above the plane of the circle such that FO is neither perpendicular to that plane nor equal to the radius of the circle. Draw FG perpendicular to the plane of the circle and let ADG be the diameter through G . Join AF , DF , and bisect the angle AFD by the straight line FC meeting AD in C . Through C draw BE perpendicular to AD , and let the tangents at B , E meet AG produced in K . Then Pappus proves that C (not O) is the centre of the apparent ellipse, that AD , BE are its major and minor axes respectively, that the ordinates to AD are parallel to BE both really and apparently, and that the ordinates to BE will pass through K but will appear to be parallel to AD . Thus in the figure, C being the centre of the apparent ellipse,



it is proved that, if LCM is any straight line through C , LC is apparently equal to CM (it is practically assumed—a proposition proved later in Book VII, Prop. 156—that, if LK meet the circle again in P , and if PM be drawn perpendicular to AD to meet the circle again in M , LM passes through O).

test of apparent equality is of course that the two straight
should subtend equal angles at F .

The main points in the proof are these. The plane through
 CK is perpendicular to the planes BFE , PFM and LFR ;
the CF is perpendicular to BE , QF to PM and HF to LR ,
hence BC and CE subtend equal angles at F : so do LH , HR ,
 PQ , QM .

Since FC bisects the angle AFD , and $AC:CD = AK:KD$
(the polar property), $\angle CFK$ is a right angle. And CF is
intersection of two planes at right angles, namely AFK
 BFE , in the former of which FK lies; therefore KF is
perpendicular to the plane BFE , and therefore to FN . Since
before (by the polar property) $LN:NP = LK:KP$, it
follows that the angle LFP is bisected by FN ; hence LN , NP
apparently equal.

Again $LC:CM = LN:NP = LF:FP = LF:FM$.

Therefore the angles LFC , CFM are equal, and LC , CM
apparently equal.

Lastly $LR:PM = LK:KP = LN:NP = LF:FP$; therefore
isosceles triangles FLR , FPM are equiangular; there-
fore the angles PFM , LFR , and consequently PFQ , LFH , are
equal. Hence LP , RM will appear to be parallel to AD .

We have, based on this proposition, an easy method of
solving Pappus's final problem (Prop. 54). 'Given a circle
 DE and any point within it, to find outside the plane of
the circle a point from which the circle will have the appear-
ance of an ellipse with centre C .'

We have only to produce the diameter AD through O to the
point K of the chord BE perpendicular to AD and then, in
the plane through AK perpendicular to the plane of the circle,
to describe a semicircle on CK as diameter. Any point F on
this semicircle satisfies the condition.

Book VII. On the 'Treasury of Analysis'.

Book VII is of much greater importance, since it gives an
account of the books forming what was called the *Treasury of*
Analysis (*ἀναλυόμενος τόπος*) and, as regards those of the books
which are now lost, Pappus's account, with the hints derivable
from the large collection of lemmas supplied by him to each

book, practically constitutes our only source of information. The Book begins (p. 634) with a definition of *analysis* and *synthesis* which, as being the most elaborate Greek utterance on the subject, deserves to be quoted in full.

'The so-called *Ἀναλυόμενος* is, to put it shortly, a special body of doctrine provided for the use of those who, after finishing the ordinary Elements, are desirous of acquiring the power of solving problems which may be set them involving (the construction of) lines, and it is useful for this alone. It is the work of three men, Euclid the author of the Elements, Apollonius of Perga and Aristaeus the elder, and proceeds by way of analysis and synthesis.'

Definition of Analysis and Synthesis.

'*Analysis*, then, takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as the result of synthesis: for in analysis we assume that which is sought as if it were already done (*γεγονός*), and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backwards (*ἀνάπαλιν λύσιν*).

'But in *synthesis*, reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what before were antecedents, and successively connecting them one with another, we arrive finally at the construction of what was sought; and this we call synthesis.

'Now analysis is of two kinds, the one directed to searching for the truth and called *theoretical*, the other directed to finding what we are told to find and called *problematical*. (1) In the *theoretical* kind we assume what is sought as if it were existent and true, after which we pass through its successive consequences, as if they too were true and established by virtue of our hypothesis, to something admitted: then (a), if that something admitted is true, that which is sought will also be true and the proof will correspond in the reverse order to the analysis, but (b), if we come upon something admittedly false, that which is sought will also be false. (2) In the *problematical* kind we assume that which is propounded as if it were known, after which we pass through its

cessive consequences, taking them as true, up to something admitted: if then (a) what is admitted is possible and obtainable, that is, what mathematicians call *given*, what was originally proposed will also be possible, and the proof will correspond in the reverse order to the analysis, but if (b) it come upon something admittedly impossible, the problem also be impossible.'

This statement could hardly be improved upon except that it ought to be added that each step in the chain of inference in the analysis must be *unconditionally convertible*; that is, in the analysis we say that, if A is true, B is true, we must be sure that each statement is a necessary consequence of the other, so that the truth of A equally follows the truth of B . This, however, is almost implied by us when he says that we inquire, not what it is (namely which follows from A , but what it is (B) from which A follows, and so on.

List of works in the 'Treasury of Analysis'.

Proppus adds a list, in order, of the books forming the *τμήματα*, namely:

Euclid's *Data*, one Book, Apollonius's *Cutting-off of a ratio*, two Books, *Cutting-off of an area*, two Books, *Determinate*, two Books, *Contacts*, two Books, Euclid's *Porisms*, two Books, Apollonius's *Inclinations or Vergings* (*πεύσεις*), two Books, the same author's *Plane Loci*, two Books, and *Surface-Loci*, eight Books, Aristaeus's *Solid Loci*, five Books, Euclid's *Surface-Loci*, two Books, Eratosthenes's *On means*, two Books. There are in all thirty-three Books, the contents of which up to the *Conics* of Apollonius I have set out for your consideration, including not only the number of the propositions, the *lemmata* and the cases dealt with in each Book, but also the *axioms* as which are required; indeed I have not, to the best of my belief, omitted any question arising in the study of the *Conics* in question.'

Description of the treatises.

It follows the short description of the contents of the *Conics* down to Apollonius's *Conics*; no account is given of Aristaeus's *Solid Loci*, Euclid's *Surface-Loci* and

Eratosthenes's *On means*, nor are there any lemmas to these works except two on the *Surface-Loci* at the end of the Book.

The contents of the various works, including those of the lost treatises so far as they can be gathered from Pappus, have been described in the chapters devoted to their authors, and need not be further referred to here, except for an *addendum* to the account of Apollonius's *Conics* which is remarkable. Pappus has been speaking of the 'locus with respect to three or four lines' (which is a conic), and proceeds to say (p. 678. 26) that we may in like manner have loci with reference to five or six or even more lines; these had not up to his time become generally known, though the synthesis of one of them, not by any means the most obvious, had been worked out and its utility shown. Suppose that there are five or six lines, and that p_1, p_2, p_3, p_4, p_5 or $p_1, p_2, p_3, p_4, p_5, p_6$ are the lengths of straight lines drawn from a point to meet the five or six at given angles, then, if in the first case $p_1 p_2 p_3 = \lambda p_4 p_5 a$ (where λ is a constant ratio and a a given length), and in the second case $p_1 p_2 p_3 = \lambda p_4 p_5 p_6$, the locus of the point is in each case a certain curve given in position. The relation could not be expressed in the same form if there were more lines than six, because there are only three dimensions in geometry, although certain recent writers had allowed themselves to speak of a rectangle multiplied by a square or a rectangle without giving any intelligible idea of what they meant by such a thing (is Pappus here alluding to Heron's proof of the formula for the area of a triangle in terms of its sides given on pp. 322-3, above?). But the system of compounded ratios enables it to be expressed for any number of lines thus, $\frac{p_1}{p_2} \cdot \frac{p_3}{p_4} \dots \frac{p_n}{a}$ (or $\frac{p_{n-1}}{p_n}$) = λ . Pappus proceeds in language not very clear (p. 680. 30); but the gist seems to be that the investigation of these curves had not attracted men of light and leading, as, for instance, the old geometers and the best writers. Yet there were other important discoveries still remaining to be made. For himself, he noticed that every one in his day was occupied with the elements, the first principles and the natural origin of the subject-matter of investigation; ashamed to pursue such topics, he had himself proved propositions of much more importance and

ty. In justification of this statement and 'in order that I may not appear empty-handed when leaving the subject', I will present his readers with the following.

(Anticipation of Guldin's Theorem.)

The enunciations are not very clearly worded, but there is no doubt as to the sense.

Figures generated by a complete revolution of a plane figure about an axis are in a ratio compounded (1) of the ratio of the areas of the figures, and (2) of the ratio of the straight lines similarly drawn to (i.e. drawn to meet at the same angles) the axes of rotation from the respective centres of gravity. Figures generated by incomplete revolutions are in the ratio compounded (1) of the ratio of the areas of the figures and (2) of the ratio of the arcs described by the centres of gravity of the respective figures, the latter ratio being itself compounded of the ratio of the straight lines similarly drawn (from the respective centres of gravity to the axes of rotation) and of the ratio of the angles contained (i.e. described) about the axes of revolution by the extremities of the said straight lines (i.e. the centres of gravity).'

Now, obviously, we have the essence of the celebrated theorem commonly attributed to P. Guldin (1677-1643), *Potestas rotunda in viam rotationis ducta producit Potestatem Rotundam uno grado altiozem Potestate sive Quantitate data*.¹

Pappus adds that

These propositions, which are practically one, include any number of theorems of all sorts about curves, surfaces, and solids, all of which are proved at once by one demonstration, and include propositions both old and new, and in particular those which have been proved in the twelfth Book of these Elements.

Kultsch attributes the whole passage (pp. 680. 30-682. 20) to an interpolator, I do not know for what reason; but it seems to me that the propositions are quite beyond what could be expected from an interpolator, indeed I know of no Greek mathematician from Pappus's day onward except Pappus himself who was capable of discovering such a proposition.

¹ *Centrobaryca*, Lib. ii, chap. viii, Prop. 3. Viennae 1641.

If the passage is genuine, it seems to indicate, what is not elsewhere confirmed, that the *Collection* originally contained, or was intended to contain, twelve Books.

Lemmas to the different treatises.

After the description of the treatises forming the *Treasury of Analysis* come the collections of lemmas given by Pappus to assist the student of each of the books (except Euclid's *Data*) down to Apollonius's *Conics*, with two isolated lemmas to the *Surface-Loci* of Euclid. It is difficult to give any summary or any general idea of these lemmas, because they are very numerous, extremely various, and often quite difficult, requiring first-rate ability and full command of all the resources of pure geometry. Their number is also greatly increased by the addition of alternative proofs, often requiring lemmas of their own, and by the separate formulation of particular cases where by the use of algebra and conventions with regard to sign we can make one proposition cover all the cases. The style is admirably terse, often so condensed as to make the argument difficult to follow without some little filling-out; the hand is that of a master throughout. The only misfortune is that, the books elucidated being lost (except the *Conics* and the *Cutting-off of a ratio* of Apollonius), it is difficult, often impossible, to see the connexion of the lemmas with one another and the problems of the book to which they relate. In the circumstances, all that I can hope to do is to indicate the types of propositions included in the lemmas and, by way of illustration, now and then to give a proof where it is sufficiently out of the common.

(a) Pappus begins with Lemmas to the *Sectio rationis* and *Sectio spatii* of Apollonius (Props. 1-21, pp. 684-704). The first two show how to divide a straight line in a given ratio, and how, given the first, second and fourth terms of a proportion between straight lines, to find the third term. The next section (Props. 3-12 and 16) shows how to manipulate relations between greater and less ratios by transforming them, e.g. *componendo*, *convertendo*, &c., in the same way as Euclid transforms *equal ratios* in Book V; Prop. 16 proves that, according as $a:b >$ or $< c:d$, $ad >$ or $< bc$. Props.

deal with three straight lines a, b, c in geometrical
 session, showing how to mark on a straight line containing
 as segments (including the whole among 'segments'),
 as equal to $a + c \pm 2\sqrt{ac}$; the lengths are of course equal
 $-c \pm 2b$ respectively. These lemmas are preliminary to
 problem (Prop. 21). Given two straight lines AB, BC
 ng between A and B), to find a point D on BA produced
 that $BD:DA = CD:(AB + BC - 2\sqrt{AB \cdot BC})$. This is,
 urse, equivalent to the quadratic equation $(a+x):x$
 $-c+x):(a+c-2\sqrt{ac})$, and, after marking off AE along
 equal to the fourth term of this proportion, Pappus solves
 equation in the usual way by application of areas.

Lemmas to the 'Determinate Section' of Apollonius.

The next set of Lemmas (Props. 22-64, pp. 704-70) belongs
 to the *Determinate Section* of Apollonius. As we have seen
 80-1, above), this work seems to have amounted to
 ory of *Involution*. Whether the application of certain
 opus's lemmas corresponded to the conjecture of Zeuthen
 t, we have at all events in this set of lemmas some
 kable applications of 'geometrical algebra'. They may
 ided into groups as follows

ops. 22, 25, 29

In the figure $AD \cdot DC = BD \cdot DE$, then

$$BD:DE = AB \cdot BC:AE \cdot EC.$$



proofs by proportions are not difficult. Prop. 29 is an
 ative proof by means of Prop. 26 (see below). The
 aic equivalent may be expressed thus: if $ax = by$, then

$$\frac{b}{y} = \frac{(a+b)(b+x)}{(a+y)(x+y)}.$$

ops. 30, 32, 34.

In the same figure $AD \cdot DE = BD \cdot DC$, then

$$BD:DC = AB \cdot BE:EC \cdot CA.$$

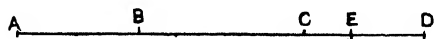
(Props. 31, 33 respectively). The algebraic equivalent is stated thus: if $ax = by$, then $\frac{b}{y} = \frac{(a+b)(b-x)}{(x+y)(a-y)}$.

III. Props. 35, 36.

If $AB \cdot BE = CB \cdot BD$, then $AB : BE = DA : AC$ and $CB : BD = AC : CE : AD : DE$, results equivalent to the following: if $ax = by$, then

$$\frac{a}{x} = \frac{(a-y)(a-b)}{(b-x)(y-x)} \quad \text{and} \quad \frac{b}{y} = \frac{(a-b)(b-x)}{(a-y)(y-x)}$$

IV. Props. 23, 24, 31, 57, 58.

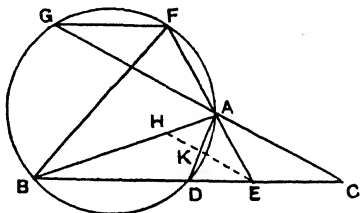


If $AB = CD$, and E is any point in CD ,

$$AC \cdot CD = AE \cdot ED + BE \cdot EC,$$

and similar formulae hold for other positions of E between B and C , $AC \cdot CD = AE \cdot ED - BE \cdot EC$. If E is on AD produced, $BE \cdot EC = AE \cdot ED + BD \cdot DC$.

V. A small group of propositions relate to a triangle with two straight lines AD , AE drawn from the vertex A to points on the base BC in accordance with one or other of the conditions (a) that the angles BAC , DAE are supplementary, (b) that the angles BAE , DAC are both right angles.



may add from Book VI, Prop. 12, (c) that the angles EAC are equal. The theorems are:

- In case (a) $BC \cdot CD : BE \cdot ED = CA^2 : AE^2$,
 „ (b) $BC \cdot CE : BD \cdot DE = CA^2 : AD^2$,
 „ (c) $DC \cdot CE : EB \cdot BD = AC^2 : AB^2$.

st (Prop. 26) because it is a case of *theoretical analysis* followed by *synthesis*. Describe a circle about ABD : produce BA , CA to meet the circle again in F , G , and join BF , FG .

Substituting $GC \cdot CA$ for $BC \cdot CD$ and $FE \cdot EA$ for $BE \cdot ED$, we have to inquire whether $GC \cdot CA : CA^2 = FE \cdot EA : AE^2$,

i.e. whether $GC : CA = FE : EA$,

i.e. whether $GA : AC = FA : AE$,

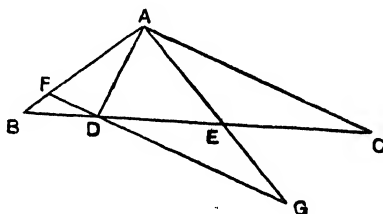
i.e. whether the triangles GAF , CAE are similar or, in other words, whether GF is parallel to BC .

But GF is parallel to BC , because, the angles BAC , DAE being supplementary, $\angle DAE = \angle GAB = \angle GFB$, while at the same time $\angle DAE = \text{suppt. of } \angle FAD = \angle FBD$.

The synthesis is obvious.

An alternative proof (Prop. 27) dispenses with the circle, and only requires EKH to be drawn parallel to CA to meet AB , AD in H , K .

Similarly (Prop. 28) for case (b) it is only necessary to draw G through D parallel to AC meeting BA in F and AE produced in G .



Then, $\angle FAG$, $\angle ADF$ ($= \angle DAC$) being both right angles, $D \cdot DG = DA^2$.

Therefore $CA^2 : AD^2 = CA^2 : FD \cdot DG = (CA : FD) \cdot (CA : DG)$
 $= (BC : BD) \cdot (CE : DE)$
 $= BC \cdot CE : BD \cdot DE$.

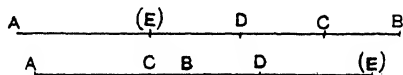
In case (c) a circle is circumscribed to ADE cutting AB in F and AC in G . Then, since $\angle FAD = \angle GAE$, the arcs DF , EG are equal and therefore FG is parallel to DE . The proof is the same as that of case (a).

VI. Props. 37, 38.

If $AB:BC = AD^2:DC^2$, whether AB be greater or than AD , then

$$AB \cdot BC = BD^2.$$

[E in the figure is a point such that $ED = CD$.]



The algebraical equivalent is: If $\frac{a}{c} = \frac{(a \pm b)^2}{(b \pm c)^2}$, then ac

These lemmas are subsidiary to the next (Props. 39, being used in the first proofs of them.

Props. 39, 40 prove the following:

If $ACDEB$ be a straight line, and if

$$BA \cdot AE : BD \cdot DE = AC^2 : CD^2,$$

then

$$AB \cdot BD : AE \cdot ED = BC^2 : CE^2;$$

if, again,

$$AC \cdot CB : AE \cdot EB = CD^2 : DE^2,$$

then

$$EA \cdot AC : CB \cdot BE = AD^2 : DB^2.$$

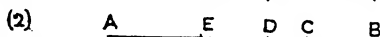
If $AB = a$, $BC = b$, $BD = c$, $BE = d$, the algebraic equivalents are the following.

$$\text{If } \frac{a(a-d)}{c(c-d)} = \frac{(a-b)^2}{(b-c)^2}, \text{ then } \frac{ac}{(a-d)(c-d)} = \frac{b^2}{(b-d)^2};$$

$$\text{and if } \frac{(a-b)b}{(a-d)d} = \frac{(b-c)^2}{(c-d)^2}, \text{ then } \frac{(a-d)(a-b)}{bd} = \frac{(a-c)^2}{c^2}.$$

VII. Props. 41, 42, 43.

If $AD \cdot DC = BD \cdot DE$, suppose that in Figures (1) and



The algebraical equivalents for Figures (1) and (2) respectively may be written (if $a = AD$, $b = DC$, $c = BD$, $d = DE$):

$$\begin{aligned}\text{If } ab = cd, \text{ then } (a \pm d + c \pm b) \quad a &= (a + c) (a \pm d), \\ &(a \pm d + c \pm b) \quad b = (c \pm b) (b + d), \\ &(a \pm d + c \pm b) \quad c = (c + a) (c \pm b), \\ &(a \pm d + c \pm b) \quad d = (a \pm d) (d + b).\end{aligned}$$

Figure (3) gives other varieties of sign. Troubles about sign can be avoided by measuring all lengths in one direction from an origin O outside the line. Thus, if $OA = a$, $OB = b$, &c., the proposition may be as follows:

$$\begin{aligned}\text{If } (d-a)(d-c) &= (b-d)(e-d) \text{ and } k = e-a+b-c, \\ \text{then } k(d-a) &= (b-a)(e-a), \quad k(d-c) = (b-c)(e-c), \\ k(b-d) &= (b-a)(b-c) \text{ and } k(e-d) = (e-a)(e-c).\end{aligned}$$

VIII. Props. 45-56.

More generally, if $AD \cdot DC = BD \cdot DE$ and $k = AE \pm BC$, then, if F be any point on the line, we have, according to the position of F in relation to A, B, C, D, E ,

$$\pm AF \cdot FC \pm EF \cdot FB = k \cdot DF.$$

Algebraically, if $OA = a$, $OB = b \dots OF = x$, the equivalent is: If $(d-a)(d-c) = (b-d)(e-d)$, and $k = (e-a) + (b-c)$, then

$$(x-a)(x-c) + (x-e)(b-x) = k(x-d).$$

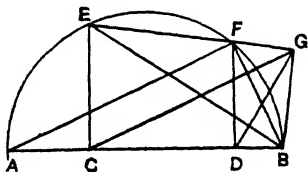
By making $x = a, b, c, e$ successively in this equation, we obtain the results of Props. 41-3 above.

IX. Props. 59-64.

In this group Props. 59, 60, 63 are lemmas required for the remarkable propositions (61, 62, 64) in which Pappus investigates 'singular and minimum' values of the ratio

but also what the value is, for three different positions of the line, in relation to the four given points.

I will give, as an illustration, the first case, on a line of great elegance. It depends on the following *Lemma*. Let AB be a diameter of a semicircle, C, D any two points on AB , and CE, DF being perpendicular to AB , let EF be drawn



produced, and let BG be drawn perpendicular to AB . We shall prove that

$$CB \cdot BD = BG^2,$$

$$AC \cdot DB = FG^2,$$

$$AD \cdot BC = EG^2.$$

Join GC, GD, FB, EB, AF .

(1) Since the angles at G, D are right, F, G, B, D are concyclic. Similarly E, G, B, C are concyclic.

Therefore

$$\angle BGD = \angle BFD$$

$$= \angle FAB$$

$$= \angle FEB, \text{ in the same segment of the circle } EGBF$$

$$= \angle GCB, \text{ in the same segment of the circle } EGBF$$

And the triangles GCB, DGB also have the angle G common; therefore they are similar, and $CB : BG = BG : BD$

or
$$CB \cdot BD = BG^2.$$

(2) We have $AB \cdot BD = BF^2$;

therefore, by subtraction, $AC \cdot DB = BF^2 - BG^2$

(3) Similarly $AB \cdot BC = BE^2$;

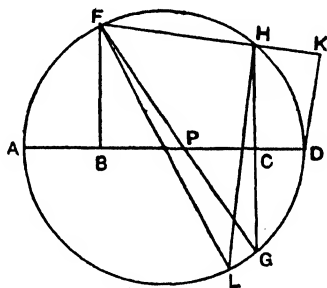
Now suppose (A, D) , (B, C) to be two point-pairs on a straight line, and let P , another point on it, be determined by the relation

$$AB \cdot BD : AC \cdot CD = BP^2 : CP^2;$$

then, says Pappus, the ratio $AP \cdot PD : BP \cdot PC$ is singular and a minimum, and is equal to

$$AD^2 : (\sqrt{AC \cdot BD} - \sqrt{AB \cdot CD})^2.$$

On AD as diameter draw a circle, and draw BF , CG perpendicular to AD on opposite sides.



Then, by hypothesis, $AB \cdot BD : AC \cdot CD = BP^2 : CP^2$;
therefore $BF^2 : CG^2 = BP^2 : CP^2$,

or $BF : CG = BP : CP$,

whence the triangles FBP , GCP are similar and therefore equiangular, so that FPG is a straight line.

Produce GC to meet the circle in H , join FH , and draw DK perpendicular to FH produced. Draw the diameter FL and join LH .

Now, by the lemma, $FK^2 = AC \cdot BD$, and $HK^2 = AB \cdot CD$;
therefore $FH = FK - HK = \sqrt{AC \cdot BD} - \sqrt{AB \cdot CD}$.

Since, in the triangles FHL , PCG , the angles at H , C are right and $\angle FLH = \angle PGC$, the triangles are similar, and

$$GP : PC = FL : FH = AD : FH$$

$$AD : (\sqrt{AC \cdot BD} - \sqrt{AB \cdot CD})$$

Therefore

$$AP \cdot PD : BP \cdot PC = AD^2 : \{ \sqrt{(AC \cdot BD)} - \sqrt{(AB \cdot CD)} \}^2$$

The proofs of Props. 62 and 64 are different, the first being long and involved. The results are:

Prop. 62. If P is between C and D , and

$$AD \cdot DB : AC \cdot CB = DP^2 : PC^2,$$

then the ratio $AP \cdot PB : CP \cdot PD$ is singular and a mean ratio and is equal to $\{ \sqrt{(AC \cdot BD)} + \sqrt{(AD \cdot BC)} \}^2 : DC^2$.

Prop. 64. If P is on AD produced, and

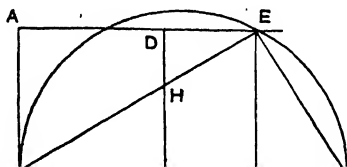
$$AB \cdot BD : AC \cdot CD = BP^2 : CP^2,$$

then the ratio $AP \cdot PD : BP \cdot PC$ is singular and a mean ratio and is equal to $AD^2 : \{ \sqrt{(AC \cdot BD)} + \sqrt{(AB \cdot CD)} \}^2$.

(γ) *Lemmas on the Νεύσεις of Apollonius.*

After a few easy propositions (e.g. the equivalent proposition that, if $ax + x^2 = by + y^2$, then, according to whether $a < b$, $a + x > b + y$ or $a + x < b + y$), Pappus gives (Prop. 65) a lemma leading to the solution of the *νέυσις* with reference to the rhombus (see pp. 190–2, above), and after that the solution by one Heraclitus of the same problem with reference to a square (Props. 71, 72, pp. 780–4). The problem is, given a square $ABCD$, to draw through B a straight line, meeting CD in H and AD produced in E , such that HE is equal to the side length.

The solution depends on a lemma to the effect that if a straight line BHE through B meets CD in H and AD produced in E , then



draw EG perpendicular to BF .

Then the triangles BCH , EGF are similar and (since EG) equal in all respects; therefore $EF = BH$.

$$BF^2 = BE^2 + EF^2,$$

$$BC \cdot BF + BF \cdot FC = BH \cdot BE + BE \cdot EH + EF^2.$$

But, the angles HCF , HEF being right, H , C , F , E are concyclic, and $BC \cdot BF = BH \cdot BE$.

Therefore, by subtraction,

$$\begin{aligned} BF \cdot FC &= BE \cdot EH + EF^2 \\ &= BE \cdot EH + BH^2 \\ &= BH \cdot HE + EH^2 + BH^2 \\ &= EB \cdot BH + EH^2 \\ &= FB \cdot BC + EH^2. \end{aligned}$$

Taking away the common part, $BC \cdot CF$, we have

$$CF^2 = BC^2 + EH^2.$$

Now suppose that we have to draw BHE through B in a way that $HE = k$. Since BC , EH are both given, we only to determine a length x such that $x^2 = BC^2 + k^2$, take BC to F so that $CF = x$, draw a semicircle on BF as center, produce AD to meet the semicircle in E , and join BE is thus the straight line required.

Prop. 73 (pp. 784–6) proves that, if D be the middle point of the base of an isosceles triangle ABC , then BC is the least of all the straight lines through D terminated by straight lines AB , AC , and the nearer to BC is shorter than the more remote.

It follows a considerable collection of lemmas mostly proving the equality of certain intercepts made on straight lines through one extremity of the diameter of one of two

In the first figure (Prop. 83), ABC , DEF being the circles, $BEKC$ is any straight line through C cutting FG is made equal to AD ; AB is joined; GH is drawn perpendicular to BK produced. It is required to prove

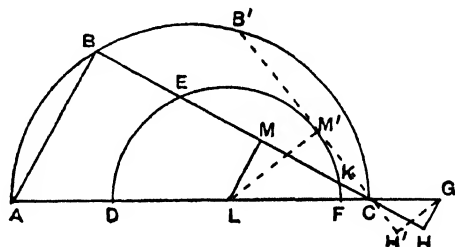


FIG. 1.

$BE = KH$. (This is obvious when from L , the centre of semicircle DEF , LM is drawn perpendicular to BK .) M and M' coincide in the point M' of the semicircle so that $B'M'$ is a tangent, then $B'M' = M'H'$ (Props. 83, 84).

In the second figure (Prop. 91) D is the centre of semicircle ABC and is also the extremity of the diameter of the semicircle DEF . If $BEGF$ be any straight line

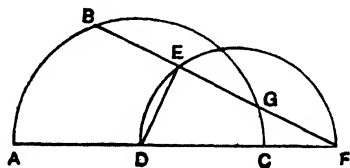


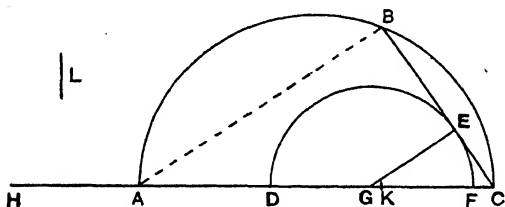
FIG. 2.

F cutting both semicircles, $BE = EG$. This is clear, since BE is perpendicular to BG .

The only problem of any difficulty in this section is Prop. 85 (p. 796). Given a semicircle ABC on the diameter AC and a point D on the diameter, to draw a semicircle with its diameter along DC such that a line CEB be drawn touching it at E and meeting the arc of semicircle ABC in B , BE shall be equal to AD .

The problem is reduced to a problem contained in Apollonius's *Determinate Section* thus.

Suppose the problem solved by the semicircle DEF , BE being equal to AD . Join E to the centre G of the semicircle



DEF . Produce DA to H , making HA equal to AD . Let K be the middle point of DC .

Since the triangles ABC , GEC are similar,

$$\begin{aligned}
 AG^2 : GC^2 &= BE^2 : EC^2 \\
 &= AD^2 : EC^2, \text{ by hypothesis,} \\
 &= AD^2 : GC^2 - DG^2 \text{ (since } DG = GE) \\
 &= AG^2 - AD^2 : DG^2 \\
 &= HG \cdot DG : DG^2 \\
 &= HG : DG.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 HG : DG &= AD^2 : GC^2 - DG^2 \\
 &= AD^2 : 2 DC \cdot GK.
 \end{aligned}$$

Take a straight line L such that $AD^2 = L \cdot 2 DC$;

therefore $HG : DG = L : GK$,

or $HG \cdot GK = L \cdot DG$.

Therefore, given the two straight lines HD , DK (or the three points H , D , K on a straight line), we have to find a point G between D and K such that

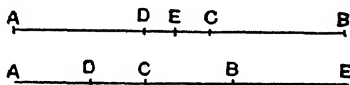
The other problem (Prop. 117, pp. 848–50), is, as we have seen, equivalent to the following: *Given a circle and three points D, E, F in a straight line external to it, to inscribe in the circle a triangle ABC such that its sides pass severally through the three points D, E, F .* For the solution, see pp. 182–4, above.

(ε) The Lemmas to the *Plane Loci* of Apollonius (Props. 119–26, pp. 852–64) are mostly propositions in geometrical algebra worked out by the methods of Eucl., Books II and VI. We may mention the following:

Prop. 122 is the well-known proposition that, if D be the middle point of the side BC in a triangle ABC ,

$$BA^2 + AC^2 = 2(AD^2 + DC^2).$$

Props. 123 and 124 are two cases of the same proposition, the enunciation being marked by an expression which is also found in Euclid's *Data*. Let $AB:BC$ be a given ratio, and



let the rectangle $CA \cdot AD$ be given; then, if BE is a mean proportional between DB, BC , 'the square on AE is greater by the rectangle $CA \cdot AD$ than in the ratio of AB to BC to the square on EC ', by which is meant that

$$AE^2 = CA \cdot AD + \frac{AB}{BC} \cdot EC^2,$$

or

$$(AE^2 - CA \cdot AD) : EC^2 = AB : BC.$$

The algebraical equivalent may be expressed thus (if $AB = a$, $BC = b$, $AD = c$, $BE = x$):

If $x = \sqrt{(a-c)b}$, then $\frac{(a \mp x)^2 - (a-b)c}{(x \mp b)^2} = \frac{a}{b}.$

This is equivalent to the general relation between points on a straight line discovered by Simson and there wrongly known as Stewart's theorem:

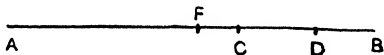
$$AD^2 \cdot BC + BD^2 \cdot CA + CD^2 \cdot AB + BC \cdot CA \cdot AB = 0.$$

(Simson discovered this theorem for the more general where D is a point outside the line ABC .)

An algebraical equivalent is the identity

$$(d-a)^2(b-c) + (d-b)^2(c-a) + (d-c)^2(a-b) + (b-c)(c-a)(a-b)$$

Pappus's proof of the last-mentioned lemma is perfectly worth giving.



C, D being two points on the straight line AB , take point F on it such that

$$FD : DB = AC : CB.$$

Then $FB : BD = AB : BC,$

and $(AB - FB) : (BC - BD) = AB : BC,$

or $AF : CD = AB : BC,$

and therefore

$$AF \cdot CD : CD^2 = AB : BC.$$

From (1) we derive

$$\frac{AC}{CB} \cdot DB^2 = FD \cdot DB,$$

and from (2)

$$\frac{AB}{BC} \cdot CD^2 = AF \cdot CD.$$

that $AD \cdot DC + FD \cdot DB = AC \cdot DB + AF \cdot CD$,

i.e. (if $AF \cdot CD$ be subtracted from each side)

that $FD \cdot DC + FD \cdot DB = AC \cdot DB$,

or $FD \cdot CB = AC \cdot DB$:

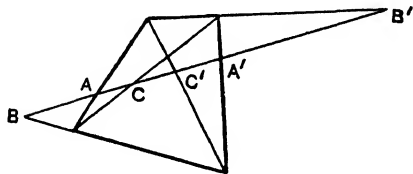
which is true, since, by (1) above, $FD : DB = AC : CB$.

(§). *Lemmas to the 'Porisms' of Euclid.*

The 38 Lemmas to the *Porisms* of Euclid form an important collection which, of course, has been included in one form or other in the 'restorations' of the original treatise. Chasles¹ in particular gives a classification of them, and we cannot do better than use it in this place: '23 of the Lemmas relate to rectilineal figures, 7 refer to the harmonic ratio of four points, and 8 have reference to the circle.

'Of the 23 relating to rectilineal figures, 6 deal with the quadrilateral cut by a transversal; 6 with the equality of the anharmonic ratios of two systems of four points arising from the intersections of four straight lines issuing from one point with two other straight lines; 4 may be regarded as expressing a property of the hexagon inscribed in two straight lines; 2 give the relation between the areas of two triangles which have two angles equal or supplementary; 4 others refer to certain systems of straight lines; and the last is a case of the problem of the *Cutting-off of an area*.'

The lemmas relating to the quadrilateral and the transversal are 1, 2, 4, 5, 6 and 7 (Props. 127, 128, 130, 131, 132, 133). Prop. 130 is a general proposition about any transversal



whatever, and is equivalent to one of the equations by which we express the involution of six points. If $A, A'; B, B'; C, C'$ be the points in which the transversal meets the pairs of

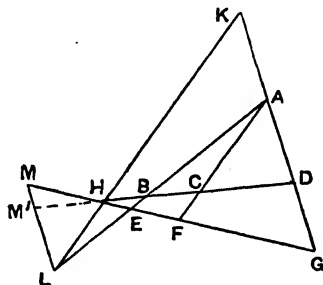
¹ Chasles, *Les trois livres de Porismes d'Euclide*, Paris, 1860, pp. 74 sq.

result is equivalent to

$$\frac{AB \cdot B'C}{A'B' \cdot BC'} = \frac{CA}{C'A'}.$$

Props. 127, 128 are particular cases in which the transversal is parallel to a side; in Prop. 131 the transversal passes through the points of concurrence of opposite sides, the result is equivalent to the fact that the two diagonals divide into proportional parts the straight line joining the points of concurrence of opposite sides; Prop. 132 is the particular case of Prop. 131 in which the line joining the points of concurrence of opposite sides is parallel to a diagonal; in Prop. 133 the transversal passes through one only of the points of concurrence of opposite sides and is parallel to a diagonal, the result is $CA^2 = CB \cdot CB'$.

Props. 129, 136, 137, 140, 142, 145 (Lemmas 3, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19) establish the equality of the anharmonic ratios of four straight lines issuing from a point determined by two transversals; but both transversals are supposed to be parallel to the same line from the same point on one of the four straight lines.



AB, AC, AD be cut by transversals $HBCD, HEFC$ required to prove that

$$\frac{HE \cdot FG}{HG \cdot EF} = \frac{HB \cdot CD}{HD \cdot BC}.$$

Pappus gives (Prop. 129) two methods of proof which are practically equivalent. The following is the proof 'by pound ratios'.

Draw HK parallel to AF meeting DA and AE in

duced in M .

$$\text{Then } \frac{HE \cdot FG}{HG \cdot EF} = \frac{HE}{EF} \cdot \frac{FG}{HG} = \frac{LH}{AF} \cdot \frac{AF}{HK} = \frac{LH}{HK}.$$

In exactly the same way, if DH produced meets LM in M' we prove that

$$\frac{HB \cdot CD}{HD \cdot BC} = \frac{LH}{HK}.$$

$$\text{Therefore } \frac{HE \cdot FG}{HG \cdot EF} = \frac{HB \cdot CD}{HD \cdot BC}.$$

(The proposition is proved for $HBCD$ and any other transversal not passing through H by applying our proposition twice, as usual.)

Props. 136, 142 are the reciprocal; Prop. 137 is a particular case in which one of the transversals is parallel to one of the straight lines, Prop. 140 a reciprocal of Prop. 137, Prop. 145 another case of Prop. 129.

The Lemmas 12, 13, 15, 17 (Props. 138, 139, 141, 143) are equivalent to the property of the hexagon inscribed in two straight lines, viz. that, if the vertices of a hexagon are situate, three and three, on two straight lines, the points of concurrence of opposite sides are in a straight line; in Props. 138, 141 the straight lines are parallel, in Props. 139, 143 not parallel.

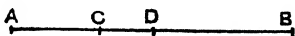
Lemmas 20, 21 (Props. 146, 147) prove that, when one angle of one triangle is equal or supplementary to one angle of another triangle, the areas of the triangles are in the ratios of the rectangles contained by the sides containing the equal or supplementary angles.

The seven Lemmas 22, 23, 24, 25, 26, 27, 34 (Props. 148-53 and 160) are propositions relating to the segments of a straight line on which two intermediate points are marked. Thus:

Props. 148, 150.

If C, D be two points on AB , then

$$(a) \text{ if } 2AB \cdot CD = CB^2, AD^2 = AC^2 + DB^2;$$



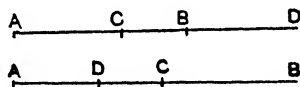
$$(b) \text{ if } 2AC \cdot BD = CD^2, AB^2 = AD^2 + CB^2.$$

Props. 149, 151.

If $AB \cdot BC = BD^2$,

then $(AD \pm DC) BD = AD \cdot DC$,

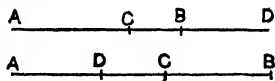
$$(AD \pm DC) BC = DC^2,$$



and $(AD \pm DC) BA = AD^2$.

Props. 152, 153.

If $AB : BC = AD^2 : DC^2$, then $AB \cdot BC = BA$



Prop. 160.

If $AB : BC = AD : DC$, then, if E be the middle p

$$BE \cdot ED = EC^2,$$

$$BD \cdot DE = AD \cdot DC,$$

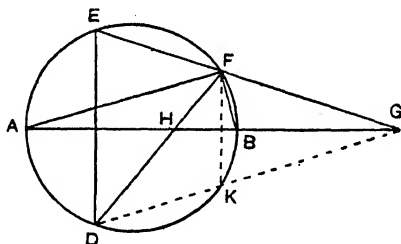
$$EB \cdot BD = AB \cdot BC.$$



The Lemmas about the circle include the harmonics of the pole and polar, whether the pole is external to the circle (Prop. 154) or internal (Prop. 161). Prop. 160 is a problem, Given a segment of a circle on AB as base, draw straight lines AC , BC to the segment in a given ratio to another.

Prop. 156 is one which Pappus has already used in the *Collection*. It proves that the straight line from the extremities of a chord (DE) to any point

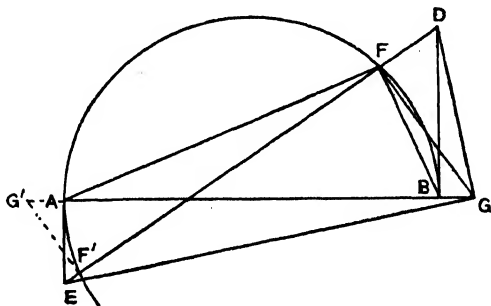
Since AB bisects DE perpendicularly, $(\text{arc } AE) = (\text{arc } AD)$ and $\angle EFA = \angle AFD$, or AF bisects the angle EFD .



Since the angle AFB is right, FB bisects $\angle HFG$, the supplement of $\angle EFD$.

Therefore (Eucl. VI. 3) $GB : BH = GF : FH = GA : AH$,
and, alternately and inversely, $AH : HB = AG : GB$.

Prop. 157 is remarkable in that (without any mention of a conic) it is practically identical with Apollonius's *Conics* III. 45 about the foci of a central conic. Pappus's theorem is as follows. Let AB be the diameter of a semicircle, and



from A, B let two straight lines AE, BD be drawn at right angles to AB . Let any straight line DE meet the two perpendiculars in D, E and the semicircle in F . Further, let FG be drawn at right angles to DE , meeting AB produced in G .

But, since A, E, G, F are concyclic, $\angle AFE = \angle AGE$

Therefore $\angle BDG = \angle AGE$;

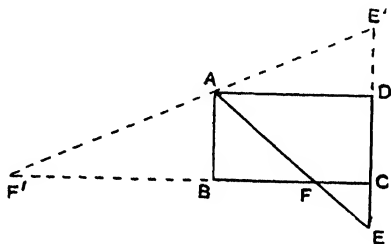
and the right-angled triangles DBG, GAE are similar

Therefore $AG : AE = BD : GB$,

or $AG \cdot GB = AE \cdot DB$.

In Apollonius G and the corresponding point produced which is obtained by drawing $F'G'$ perpendicular to ED (where DE meets the circle again in F') are the foci of a central conic (in this case a hyperbola), and $AE \cdot DB$ is the square of the semi-latus rectum tangent to the conic; the rectangle $AE \cdot DB$ is of course equal to the square on half the conjugate axis.

(η) The Lemmas to the *Conics* of Apollonius (pp. 170-171) do not call for any extended notice. There are a large number of propositions in geometrical algebra of the type relating to the segments of a straight line marked by a series of points on it; propositions about lines divided in harmonic or other special segments and about similar figures; two propositions relating to the construction of a hyperbola (Prop. 197 and 198) and a proposition (208) proving that two hyperbolas with the same asymptotes do not meet one another. The last two propositions (221, 222) are equivalent to an obvious



metrical formula. Let $ABCD$ be a rectangle, a straight line through A meet DC produced in E (produced if necessary) in F .

Then $EA \cdot AF = ED \cdot DC + CB \cdot BF$.

$$= ED^2 + BC^2 + CD^2 + BF^2.$$

Also $EA^2 + AF^2 = EF^2 + 2EA \cdot AF.$

Therefore

$$\begin{aligned} 2EA \cdot AF &= EA^2 + AF^2 - EF^2 \\ &= ED^2 + BC^2 + CD^2 + BF^2 - EF^2 \\ &= (ED^2 + CD^2) + (BC^2 + BF^2) - EF^2 \\ &= EC^2 + 2ED \cdot DC + CF^2 + 2CB \cdot BF - EF^2 \\ &= 2ED \cdot DC + 2CB \cdot BF; \end{aligned}$$

i.e. $EA \cdot AF = ED \cdot DC + CB \cdot BF.$

This is equivalent to $\sec \theta \operatorname{cosec} \theta = \tan \theta + \cot \theta.$

The algebraical equivalents of some of the results obtained by the usual geometrical algebra may be added.

Props. 178, 179, 192-4.

$$(a + 2b)a + (b + x)(b - x) = (a + b + x)(a + b - x).$$

Prop. 195. $4a^2 = 2\{ (a - x)(a + x) + (a - y)(a + y) + x^2 + y^2 \}.$

Prop. 196.

$$(a + b - x)^2 + (a + b + x)^2 = (x - b)^2 + (x + b)^2 + 2(a + 2b)a.$$

Props. 197, 199, 198.

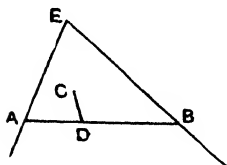
$$\left. \begin{array}{l} \text{If} \quad (x + y + a)a + x^2 = (a + x)^2, \\ \text{or if} \quad (x + y + a)a + x^2 = (a + y)^2, \\ \text{or if} \quad (x + y - a)a + (x - a)^2 = y^2, \end{array} \right\} \text{ then } x = y.$$

Props. 200, 201. If $(a + b)x = b^2$, then $\frac{2b + a}{a} = \frac{b + x}{b - x}$ and $(2b + a)a = (a + b)(a + b - x).$

Prop. 207. If $(a + b)b = 2a^2$, then $a = b.$

(θ) The two Lemmas to the *Surface-Loci* of Euclid have already been mentioned as significant. The first has the appearance of being a general enunciation, such as Pappus

is fond of giving, to cover a class of propositions, the enunciation may be translated as follows: 'If AB a straight line, and CD a straight line parallel to a straight line in a given position, and if the ratio $AD \cdot DB : DC^2$ be given, the locus of the point E lies on a conic section. If now AB be no longer in a given position, and the points A, B are no longer given in position (respectively) on straight lines AE, EB given in position, the point C raised above (the plane containing AE, EB) lies on a surface given in position. And this was proved by Pappus.



was the first to explain this, and his interpretation only a very slight change in the Greek text, substituting *εὐθείαις* for *εὐθείαις* γέννηται δὲ πρὸς θέσει *εὐθείαις*. It is not clear whether, when the lines AE, EB are given, it is to be given in position, it

in length. If it is given in length and A, B move on straight lines AE, EB respectively, the surface which is the locus of the point E is a complicated one such as Euclid would hardly have been in a position to investigate. But two possibilities are indicated which he may have discussed, (1) that in which the lines AE, EB move always parallel to themselves and varies in length, (2) that in which the two lines on which A, B move are parallel instead of meeting at a point. The loci of the point E in these cases would of course be a cone and a cylinder respectively.

The second Lemma is still more important, since it is the first statement on record of the focus-directrix property of the three conic sections. The proof, after Pappus, is set out above (pp. 119–21):

(1) *An unallocated Lemma.*

Book VII ends (pp. 1016–18) with a lemma which is given under any particular treatise belonging to the *Book of Analysis*, but is simply called 'Lemma to the 2nd Proposition'. If ABC be a triangle right-angled at B , and

Historical preface.

It begins with an interesting preface on the nature of theoretical mechanics, as distinct from the mechanical or industrial, to be regarded as a mathematical science. Archimedes, Philon, Heron of Alexandria are mentioned as the principal exponents of the science, while Carpus is also mentioned as having applied geometry to the (practical) arts'.

The date of Carpus is uncertain, though it is probable he came after Geminus; the most likely date seems to be the first or second century A.D. Simplicius gives the authority of Iamblichus for the statement that Carpus squared the circle by means of a certain curve, which he simply called a curve generated by a double motion.¹ Proclus calls him a writer on mechanics (*ὁ μηχανικός*), and quotes from his work on Astronomy some remarks about the relation of problems and theorems and the 'priority in the former.'² Proclus also mentions him as having shown that an angle belongs to the category of *quantity* (*ποσότης*); it represents a sort of 'distance' between the two lines forming it, this distance being 'extended one way' (*ἐκτετακμένη*), though in a different sense from that in which a line is extended one way, so that Carpus's view appeared to be the greatest possible paradox'³; Carpus seems in reality to have been anticipating the modern view of an angle as a *divergence* rather than distance, and to have measured it *in one sense* (rotationally), as distinct from one *dimension* (linearly).

Pappus tells us that Heron distinguished the theoretical part of mechanics from the practical (*χειρουργικόν*), the former being made up of geometry, arithmetic, astronomy and physics, the latter of work in architecture, carpentering and painting; the mathematician has been trained from his youth up in the *sciences* as distinct from the *arts* as practised in the said *arts* would naturally p

must perforce use the resources which practical experience in his particular art or craft gives him. Other varieties of mechanical work included by the ancients under the general term mechanics were (1) the use of the mechanical powers or devices for moving or lifting great weights by means of a small force, (2) the construction of engines of war for throwing projectiles a long distance, (3) the pumping of water from great depths, (4) the devices of 'wonder-workers' (*θαυμασιουργοί*), some depending on pneumatics (like Heron in the *Pneumatica*), some using strings, &c., to produce movements like those of living things (like Heron in 'Automata and Balancings'), some employing floating bodies (like Archimedes in 'Floating Bodies'), others using water to measure time (like Heron in his 'Water-clocks'), and lastly 'sphere-making' or the construction of mechanical imitations of the movement of the heavenly bodies with the uniform circular motion of water as the motive power. Archimedes, says Pappus, was held to be the one person who had understood the cause and the reason of all these various devices, and had applied his extraordinarily versatile genius and inventiveness to all the purposes of daily life, and yet, although this brought him unexampled fame the world over, so that his name was on every one's lips, he disdained (according to Carpus) to write any mechanical work save a tract on sphere-making, but diligently wrote all that he could in a small compass of the most advanced parts of geometry and of subjects connected with arithmetic. Carpus himself, says Pappus, as well as others applied geometry to practical arts, and with reason 'for geometry is in no wise injured, nay it is by nature capable of giving substance to many arts by being associated with them, and, so far from being injured, it may be said while itself advancing those arts, to be honoured and adorned by them in return.'

The object of the Book.

Pappus then describes the object of the Book, namely, to set out the propositions which the ancients established by geometrical methods, besides certain useful theorems discovered by himself, but in a shorter and clearer form and

in better logical sequence than his predecessors had. The sort of questions to be dealt with are (1) a comparison between the force required to move a given weight on a horizontal plane and that required to move the same weight upwards on an inclined plane, (2) the finding of two proportionals between two unequal straight lines, (3) to find a toothed wheel with a certain number of teeth, to a given diameter of, and to construct, another wheel with a given number of teeth to work on the former. Each of these things, will be clearly understood in its proper place if the principles on which the 'centrobaric doctrine' is built up are first stated. It is not necessary, he adds, to define what is meant by 'heavy' and 'light' or upward and downward motion, since these matters are discussed by Ptolemy in his *Mathematica*. The notion of the centre of gravity is so fundamental to the whole theory of mechanics that it is essential in every place to explain what is meant by the 'centre of gravity' of any body.

On the centre of gravity.

Pappus then defines the centre of gravity as 'the point within a body which is such that, if the weight be cut off to be suspended from the point, it will remain at rest in the position in which it is put'.¹ The method of determining the point by means of the intersection, first of planes, and then of straight lines, is next explained (chaps. 1, 2), and Pappus proves (Prop. 2) a proposition of some difficulty, namely, if D, E, F be points on the sides BC, CA, AB of a triangle such that

$$BD : DC = CE : EA = AF : FB,$$

then the centre of gravity of the triangle ABC is the same as the centre of gravity of the triangle DEF .

Let H, K be the middle points of BC, CA respectively. Join AH, BK . Join HK meeting DE in L .

Now, by hypothesis,

$$CE : EA = BD : DC,$$

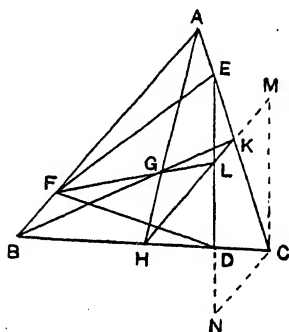
whence

$$CA : AE = BC : CD,$$

and, if we halve the antecedents,

$$AK : AE = HC : CD;$$

therefore $AK : EK = HC : HD$ or $BH : HD$,



whence, *componendo*, $CE : EK = BD : DH$. (1)

$$\begin{aligned} \text{But } AF : FB &= BD : DC = (BD : DH) \cdot (DH : DC) \\ &= (CE : EK) \cdot (DH : DC). \end{aligned} \quad (2)$$

Now, ELD being a transversal cutting the sides of the triangle KHC , we have

$$HL : KL = (CE : EK) \cdot (DH : DC). \quad (3)$$

[This is 'Menelaus's theorem'; Pappus does not, however, quote it, but proves the relation *ad hoc* in an added lemma by drawing CM parallel to DE to meet HK produced in M . The proof is easy, for $HL : LK = (HL : LM) \cdot (LM : LK) = (HD : DC) \cdot (CE : EK).$]

It follows from (2) and (3) that

$$AF \cdot FB = HL \cdot LK$$

We have next to prove that $EL = LD$.

Now [again by 'Menelaus's theorem', proved *op. cit.* by drawing CN parallel to HK to meet ED produced in N]

$$EL:LD = (EK:KC) \cdot (CH:HD).$$

But, by (1) above, $CE:EK = BD:DH$;

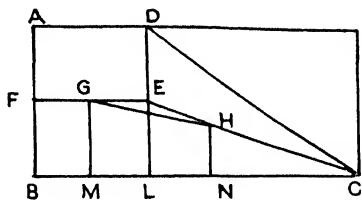
therefore $CK:KE = BH:HD = CH:HD$,

so that $(EK:KC) \cdot (CH:HD) = 1$, and therefore,

$$EL = LD.$$

It remains to prove that $FG = 2GL$, which is obvious if FG and EL are parallels, since $FG:GL = AG:GH = 2:1$.

Two more propositions follow with reference to the centre of gravity of a trapezium. The first is, Given a rectangle with AD and BC as adjacent sides, to draw from C a straight line meeting AD in a point D such that, if the trapezium $ABCD$ is hung from the point D , it will rest with AD , BC horizontal.



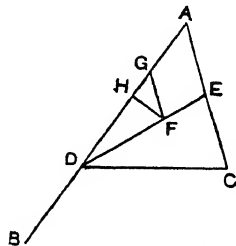
In other words, the centre of gravity must be in a line perpendicular to BC . Pappus proves by analysis that $CL^2 = 3BL^2$, so that the problem is reduced to dividing BC into parts BL , LC such that this relation holds. The latter problem is solved (Prop. 6) by taking BL as the mean proportional between BC and LC , say X , in CB such that $CX = 3XB$, describing a semicircle with BC as diameter and drawing XY at right angles to BC to meet the semicircle in Y , so that $XY^2 = \frac{3}{16} BC^2$.

with its extremities on AC , AB and so that $AC:BD$ is a given ratio, then the centre of gravity of the triangle ADC will lie on a straight line.

Take E , the middle point of AC , and F a point on DE such that $DF = 2FE$. Also let H be a point on BA such that $BH = 2HA$. Draw FG parallel to AC . Then $AG = \frac{1}{3}AD$, and $AH = \frac{1}{3}AB$; therefore $HG = \frac{1}{3}BD$.

Also $FG = \frac{2}{3}AE = \frac{1}{3}AC$. Therefore, since the ratio $AC:BD$ is given, the ratio $GH:GF$ is given.

And the angle FGH ($= A$) is given; therefore the triangle FGH is given in species, and consequently the angle GHF is given. And H is a given point. Therefore HF is a given straight line, and it contains the centre of gravity of the triangle ADC .

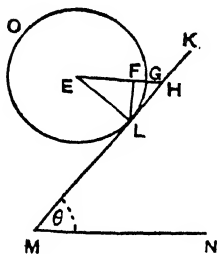


The inclined plane.

Prop. 8 is on the construction of a plane at a given inclination to another plane parallel to the horizon, and with this Pappus leaves theory and proceeds to the practical part. Prop. 9 (p. 1054. 4 sq.) investigates the problem 'Given a weight which can be drawn along a plane parallel to the horizon by a given force, and a plane inclined to the horizon at a given angle, to find the force required to draw the weight upwards on the inclined plane'. This seems to be the first or only attempt in ancient times to investigate motion on an inclined plane, and as such it is curious, though of no value.

Let A be the weight which can be moved by a force C along a horizontal plane. Conceive a sphere with weight equal to A placed in contact at L with the given inclined plane; the circle OGL represents a section of the sphere by a vertical plane

about FL as if L were the fulcrum of a lever. Now the weight A acts vertically downwards along a straight line through E . To balance it, Pappus supposes a weight attached with its centre of gravity at G .



$$\text{Then } A : B = GF : EF$$

$$= (EL - EF) : EF$$

$$[= (1 - \sin \theta) : \sin \theta, \\ \text{where } \angle KMN = \theta]$$

and, since $\angle KMN$ is given, the ratio $EF : EL$ is given; and therefore the ratio $(EL - EF) : EF$ is given; thus B is found.

Now, says Pappus, if D is the force which will move A along a horizontal plane, as C is the force which will move A along a horizontal plane, the sum of C and D will be the force required to move the sphere upwards on the inclined plane. He takes the particular case where $\theta = 60^\circ$. The $\sin \theta$ is approximately $\frac{104}{125}$ (he evidently uses $\frac{1}{2} \cdot \frac{208}{125}$ for $\frac{1}{2} \sqrt{3}$).

and

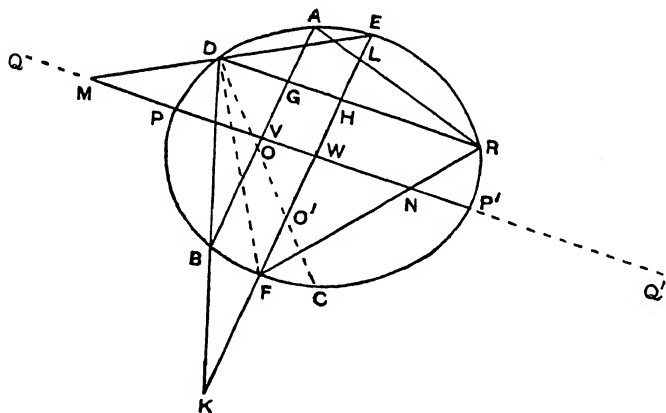
$$A : B = 16 : 104.$$

Suppose, for example, that $A = 200$ talents; then B is 13 talents. Suppose further that C is 40 man-power; then, since $D : C = B : A$, $D = 260$ man-power; and it will take $D + C$, 300 man-power, to move the weight up the plane!

Prop. 10 gives, from Heron's *Barulcus*, the machine consisting of a pulley, interacting toothed wheels, and a spiral screw working on the last wheel and turned by a hand. Pappus merely alters the proportions of the weight to the force, and of the diameter of the wheels. At the end of the chapter (pp. 1070-2) he repeats his construction for the finding of two mean proportionals.

is left intact, and let it be required to find the diameter of a circular section of the cylinder. We take any two points A, B on the surface of the fragment and by means of these we find five points on the surface all lying in one plane section, in general oblique. This is done by taking five different radii and drawing pairs of circles with A, B as centres and with each of the five radii successively. These pairs of circles with equal radii, intersecting at points on the surface, determine five points on the plane bisecting AB at right angles. The five points are then represented on any plane by triangulation.

Suppose the points are A, B, C, D, E and are such that no two of the lines connecting the different pairs are parallel.



This case can be reduced to the construction of a conic through the five points A, B, D, E, F where EF is parallel to AB . This is shown in a subsequent lemma (chap. 16).

For, if EF be drawn through E parallel to AB , and if CD meet AB in O and EF in O' , we have, by the well-known proposition about intersecting chords,

$$CO \cdot OD : AO \cdot OB = CO' \cdot O'D : EO' \cdot OF,$$

to this diameter. Then R is determined by the relation

$$RG.GD:BG.GA = RH.HD:FH.HE$$

in this way.

Join DB, RA , meeting EF in K, L respectively.

Then, by similar triangles,

$$\begin{aligned} RG.GD:BG.GA &= (RH:HL).(DH:HK) \\ &= RH.HD:KH.HL. \end{aligned}$$

Therefore, by (1), $FH.HE = KH.HL$,

whence HL is determined, and therefore L . The intersection of AL, DH determines R .

Next, in order to find the extremities P, P' of the ellipse through V, W , we draw ED, RF meeting PP' in M, N respectively.

Then, as before,

$$\begin{aligned} FW.WE:P'W.WP &= FH.HE:RH.HD, \text{ by the eq. (1)} \\ &= FW.WE:NW.WM, \text{ by similar triangles.} \end{aligned}$$

Therefore $P'W.WP = NW.WM$;

and similarly we can find the value of $P'V.VP$.

Now, says Pappus, since $P'W.WP$ and $P'V.VP$ are areas and the points V, W are given, P, P' are given. The determination of P, P' amounts (Prop. 14 follows) to the elimination of one of the points and the finding of the other by means of an equation of the second degree.

Take two points Q, Q' on the diameter such that

$$P'V.VP = WV.VQ,$$

$$P'W.WP = VW.WQ';$$

Q, Q' are thus known, while P, P' remain to be found.

$$PQ.PQ' = QV.Q'W.$$

us P can be found, and similarly P' .

The conjugate diameter is found by virtue of the relation

$$(\text{conjugate diam.})^2 : PP'^2 = p : PP'.$$

where p is the latus rectum to PP' determined by the property of the curve

$$p : PP' = AV^2 : PV.VP'.$$

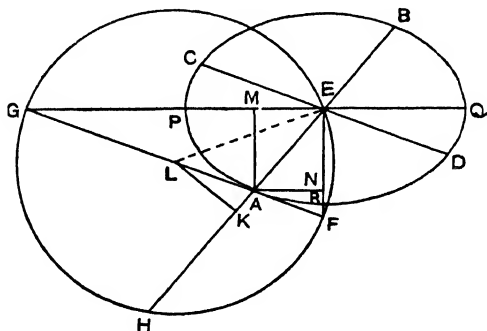
Problem, Given two conjugate diameters of an ellipse, to find the axes.

Firstly, Pappus shows (Prop. 14, chap. 17) how, when we are given two conjugate diameters, we can find the axes. The construction is as follows. Let AB, CD be conjugate diameters (AB being the greater), E the centre.

Produce EA to H so that

$$EA.AH = DE^2.$$

Through A draw FG parallel to CD . Bisect EH in K , and draw KL at right angles to EH meeting FG in L .



With L as centre, and LE as radius, describe a circle cutting the ellipse in G, F .

Join EF, EG , and from A draw AM, AN parallel to EF, EG respectively.

Then he proves that, if we join AB , AB is equal to the length of the side of the hexagon required.

Produce BC to D so that $BD = BA$, and join DA . ABA is then equilateral.

Since EB is a tangent to the segment, $AE \cdot EC = EB^2$ or $AE:EB = EB:EC$, and the triangles EAB , EBC are similar.

Therefore $BA^2:BC^2 = AE^2:EB^2 = AE:EC = 9:4$;
and $BC = \frac{2}{3}BA = \frac{2}{3}BD$, so that $BC = 2CD$.

But $CF = 2CA$; therefore $AC:CF = DC:CB$, and AD , BA are parallel.

Therefore $BF:AD = BC:CD = 2:1$, so that

$$BF = 2AD = 2AB.$$

Also $\angle FBC = \angle BDA = 60^\circ$, so that $\angle ABF = 120^\circ$, and the triangle ABF is therefore equal and similar to the required triangle NLO .

Construction of toothed wheels and indented screws.

The rest of the Book is devoted to the construction (1) of toothed wheels with a given number of teeth equal to those of a given wheel, (2) of a cylindrical helix, the *cochlias*, indented so as to work on a toothed wheel. The text is evidently defective, and at the end an interpolator has inserted extracts about the mechanical powers from Heron's *Mechanics*.

ALGEBRA: DIOPHANTUS OF ALEXANDRIA

Beginnings learnt from Egypt.

IN algebra, as in geometry, the Greeks learnt the beginnings from the Egyptians. Familiarity on the part of the Greeks with Egyptian methods of calculation is well attested. These methods are found in operation in the Heron of Alexandria and collections. (2) Psellus in the letter published in his edition of Diophantus speaks of 'the method of arithmetical calculations used by the Egyptians, by which problems in analysis are handled'; he adds details, doubtless from Anatolius, of the technical terms used for division of numbers, including the powers of the unknown. (3) The scholiast to Plato's *Charmides* 165 E says that the *λογιστική*, the science of calculation, are the so-called Egyptian methods in multiplications and divisions, and the additions and subtractions of fractions'. (4) Aristotle in the *Laws* 819 A-C says that free-born boys should practice in Egypt, learn, side by side with real life, the mathematical calculations adapted to their age, and be put into a form such as to combine amusement with instruction: problems about the distribution of garlands, the calculation of mixtures, and other problems arising in military or civil life.

'Hau'-calculations.

The Egyptian calculations here in point (and

with one unknown quantity. Examples from the Papyrus Rhind correspond to the following equations:

$$\frac{1}{7}x + x = 19,$$

$$\frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x + x = 33,$$

$$(x + \frac{2}{3}x) - \frac{1}{3}(x + \frac{2}{3}x) = 10.$$

The Egyptians anticipated, though only in an elementary form, a favourite method of Diophantus, that of the 'false supposition' or 'regula falsi'. An arbitrary assumption is made as to the value of the unknown, and the true value is afterwards found by a comparison of the result of substituting the wrong value in the original expression with the actual data. Two examples may be given. The first, from the Papyrus Rhind, is the problem of dividing 100 loaves among five persons in such a way that the shares are in arithmetical progression, and one-seventh of the sum of the first three shares is equal to the sum of the other two. If $a + 4d$, $a + 3d$, $a + 2d$, $a + d$, a be the shares, then

$$3a + 9d = 7(2a + d),$$

$$\text{or} \quad d = 5\frac{1}{2}a.$$

Ahmes says, without any explanation, 'make the difference, as it is, $5\frac{1}{2}$ ', and then, assuming $a = 1$, writes the series 23, $17\frac{1}{2}$, 12, $6\frac{1}{2}$, 1. The addition of these gives 60, and 100 is $1\frac{2}{3}$ times 60. Ahmes says simply 'multiply $1\frac{2}{3}$ times' and thus gets the correct values $38\frac{1}{3}$, $29\frac{1}{6}$, 20, $10\frac{2}{3}$, $1\frac{2}{3}$.

The second example (taken from the Berlin Papyrus 6619) is the solution of the equations

$$x^2 + y^2 = 100,$$

$$x : y = 1 : \frac{3}{4}, \text{ or } y = \frac{3}{4}x.$$

x is first assumed to be 1, and $x^2 + y^2$ is thus found to be $\frac{25}{16}$. In order to make 100, $\frac{25}{16}$ has to be multiplied by 64 or 8^2 . The true value of x is therefore 8 times 1, or 8.

Arithmetical epigrams in the Greek Anthology.

The simple equations solved in the Papyrus Rhind are just the kind of equations of which we find many examples in the

arithmetical epigrams contained in the Greek Anthology of these appear under the name of Metrodorus, a grammarian probably of the time of the Emperors Anastasius I (A.D. 491-518) and Justin I (A.D. 518-27). They were obviously collected by Metrodorus, from ancient as well as modern sources. Many of the epigrams (46 in number) lead to simple equations, and several of them are problems of dividing a number of apples or nuts among a certain number of persons. It is to say, the very type of problem mentioned by Plato. For example, a number of apples has to be determined such that if four persons out of six receive one-third, one-eighth, one-fourth and one-fifth respectively of the whole number of apples, the fifth person receives 10 apples, there is one apple left for the sixth person, i.e.

$$\frac{1}{3}x + \frac{1}{8}x + \frac{1}{4}x + \frac{1}{5}x + 10 + 1 = x.$$

Just as Plato alludes to bowls (*φιάλαι*) of different weights, there are problems in which the weights of bowls have to be found. We are thus enabled to understand the allusion of Proclus and the scholiast on *Charmides* 165 E to *φιαλῆται ἀριθμοί*, 'numbers of apples or of bowls'. It is evident from Plato's allusions that the origin of the simple algebraical problems dates back, at least, to the fifth century B.C.

The following is a classification of the problems in the *Anthology*. (1) Twenty-three are simple equations with one unknown and of the type shown above; one of them is the epigram on the age of Diophantus and certain incidents in his life (xiv. 126). (2) Twelve are easy simultaneous equations with two unknowns, like Dioph. I. 6; they can all be reduced to a simple equation with one unknown by an easy elimination. One other (xiv. 51) gives simultaneous equations in three unknowns

$$x = y + \frac{1}{3}z, \quad y = z + \frac{1}{3}x, \quad z = 10 + \frac{1}{3}y,$$

tions of this type with the same number of unknown quantities which was given by Thymaridas, an early Pythagorean, and was called the *ἐπάνθημα*, 'flower' or 'bloom' of Thymaridas (see vol. i, pp. 94-6). (3) Six more are problems of the usual type about the filling and emptying of vessels by pipes; e.g. (xiv. 130) one pipe fills the vessel in one day, a second in two and a third in three; how long will all three running together take to fill it? Another about brickmakers (xiv. 136) is of the same sort.

Indeterminate equations of the first degree.

The Anthology contains (4) two *indeterminate* equations of the first degree which can be solved in positive integers in an infinite number of ways (xiv. 48, 144); the first is a distribution of apples, $3x$ in number, into parts satisfying the equation $x - 3y = y$, where y is not less than 2; the second leads to three equations connecting four unknown quantities:

$$x + y = x_1 + y_1,$$

$$x = 2y_1,$$

$$x_1 = 3y,$$

the general solution of which is $x = 4k$, $y = k$, $x_1 = 3k$, $y_1 = 2k$. These very equations, which, however, are made determinate by assuming that $x + y = x_1 + y_1 = 100$, are solved in Dioph. I. 12.

Enough has been said to show that Diophantus was not the inventor of Algebra. Nor was he the first to solve indeterminate problems of the second degree.

Indeterminate equations of second degree before Diophantus.

Take first the problem (Dioph. II. 8) of dividing a square number into two squares, or of finding a right-angled triangle

where n is any odd number, and Plato with another form of the same sort, namely $(2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2$. (Lemma following X. 28) finds the following more general formula

$$m^2 n^2 p^2 q^2 = \left\{ \frac{1}{2} (mnp^2 + mnq^2) \right\}^2 - \left\{ \frac{1}{2} (mnp^2 - mnq^2) \right\}^2.$$

The Pythagoreans too, as we have seen (vol. i, pp. 100-101), solved another indeterminate problem, discovering, by means of the series of 'side-' and 'diameter-numbers', any number of successive integral solutions of the equations

$$2x^2 - y^2 = \pm 1.$$

Diophantus does not particularly mention this equation, but from the Lemma to VI. 15 it is clear that he knew how to find any number of solutions when one is known. Seeing that $2x^2 - 1 = y^2$ is satisfied by $x = 1$, $y = 1$, he put

$$\begin{aligned} 2(1+x)^2 - 1 &= \text{a square} \\ &= (px-1)^2, \text{ say;} \end{aligned}$$

whence

$$x = (4 + 2p)/(p^2 - 2).$$

Take the value $p = 2$, and we have $x = 4$, and $x + 1 = 5$; in this case $2 \cdot 5^2 - 1 = 49 = 7^2$. Putting $x + 5$ in place of x we can find a still higher value, and so on.

Indeterminate equations in the Heronian collection

Some further Greek examples of indeterminate analysis are now available. They come from the Constantinople manuscript (probably of the twelfth century) from which Schönemann edited the *Metrica* of Heron; they have been published and translated by Heiberg, with comments by Zeuthen.¹ Two of the problems (thirteen in number) had been published in a less complete form in Hultsch's *Heron* (*Geöponicus*, 78, 79); the others are new.

er 3 is of course only an illustration, and the problem is
 alent to the solution of the equations

$$\left. \begin{aligned} (1) \quad u + v &= n(x + y) \\ (2) \quad xy &= n \cdot uv \end{aligned} \right\}.$$

e solution given in the text is equivalent to

$$\left. \begin{aligned} x &= 2n^3 - 1, & y &= 2n^3 \\ u &= n(4n^3 - 2), & v &= n \end{aligned} \right\}.$$

then suggests that the solution may have been obtained

As the problem is indeterminate, it would be natural
 rt with some hypothesis, e.g. to put $v = n$. It would
 r from equation (1) that u is a multiple of n , say nz .
 ave then

$$x + y = 1 + z,$$

, by (2),

$$xy = n^3 z,$$

ce

$$xy = n^3(x + y) - n^3,$$

$$(x - n^3)(y - n^3) = n^3(n^3 - 1).$$

obvious solution is

$$x - n^3 = n^3 - 1, \quad y - n^3 = n^3,$$

a gives $z = 2n^3 - 1 + 2n^3 - 1 = 4n^3 - 2$, so that

$$u = nz = n(4n^3 - 2).$$

he second is a similar problem about two rectangles,
 alent to the solution of the equations

$$\left. \begin{aligned} (1) \quad x + y &= u + v \\ (2) \quad xy &= n \cdot uv \end{aligned} \right\},$$

he solution given in the text is

$$x + y = u + v = n^3 - 1, \tag{3}$$

$$u = n(4n^3 - 2), \quad v = n(n^3 - 1)$$

when equation (1) would give

$$(n-1)x = (n^2-1)u,$$

a solution of which is $x = n^2-1$, $u = n-1$.

III. The fifth problem is interesting in one respect. It is asked to find a right-angled triangle (in rational numbers) with area of 5 feet. We are told to multiply 5 by 6, the square containing 6 as a factor, e.g. 36. This makes 180, and this is the area of the triangle (9, 40, 41). Dividing the side by 6, we have the triangle required. The author is aware that the area of a right-angled triangle with sides in whole numbers is divisible by 6. If we take the Euclidean formula for a right-angled triangle, making the sides $a \cdot \frac{1}{2}(m^2-n^2)$, $a \cdot \frac{1}{2}(m^2+n^2)$, where a is any number, and m, n are numbers which are both odd or both even, the area is

$$\frac{1}{4}mn(m-n)(m+n)a^2,$$

and, as a matter of fact, the number $mn(m-n)(m+n)$ is divisible by 24, as was proved later (for another purpose) by Leonardo of Pisa.

IV. The last four problems (10 to 13) are of great interest. They are different particular cases of one problem, finding a rational right-angled triangle such that the number sum of its area and its perimeter is a given number. The author's solution depends on the following formulae. Let a, b be the perpendiculars, and c the hypotenuse, of a right-angled triangle, S its area, r the radius of the inscribed circle, and $s = \frac{1}{2}(a+b+c)$;

$$S = rs = \frac{1}{2}ab, \quad r+s = a+b, \quad c = s-r.$$

(The proof of these formulae by means of the usual method, namely that used by Heron to prove the formula

$$S = \sqrt{s(s-a)(s-b)(s-c)},$$

is easy.)

Solving the first two equations, in order to find

and b . The method employed is to take the sum of the area and the perimeter $S+2s$, separated into its two obvious factors $s(r+2)$, to put $s(r+2) = A$ (the given number), and then to separate A into suitable factors to which s and $r+2$ may be equated. They must obviously be such that sr , the area, is divisible by 6. To take the first example where $A=280$: the possible factors are 2×140 , 4×70 , 5×56 , 7×40 , 8×35 , 10×28 , 14×20 . The suitable factors in this case are $r+2=8$, $s=35$, because r is then equal to 6, and rs is a multiple of 6.

The author then says that

$$a = \frac{1}{2} [6 + 35 - \sqrt{\{(6 + 35)^2 - 8 \cdot 6 \cdot 35\}}] = \frac{1}{2} (41 - 1) = 20,$$

$$b = \frac{1}{2} (41 + 1) = 21,$$

$$c = 35 - 6 = 29.$$

The triangle is therefore (20, 21, 29) in this case. The triangles found in the other three cases, by the same method, are (9, 40, 41), (8, 15, 17) and (9, 12, 15).

Unfortunately there is no guide to the date of the problems just given. The probability is that the original formulation of the most important of the problems belongs to the period between Euclid and Diophantus. This supposition best agrees with the fact that the problems include nothing taken from the great collection in the *Arithmetica*. On the other hand, it is strange that none of the seven problems above mentioned is found in Diophantus. The five relating to rational right-angled triangles might well have been included by him; thus he finds rational right-angled triangles such that the area *plus* or *minus* one of the perpendiculars is a given number, but not the rational triangle which has a given area; and he finds rational right-angled triangles such that the area *plus* or *minus* the sum of *two* sides is a given number, but not the rational triangle such that the sum of the area and the *three* sides is a given number. The omitted problems might, it is true, have

The *geometrical* algebra of the Greeks has been in vogue all through our history from the Pythagoreans down to the present time, and no more need be said of it here except that its arithmetical application was no new thing in Diophantus. It is possible, for example, that the solution of the quadratic equations discovered first by geometry, was applied for the purpose of finding *numerical* values for the unknown as early as the fifth century, if not earlier still. In Heron the numerical solution of quadratic equations is well established, so that Diophantus was the first to treat equations algebraically. What he did was to take a step forward towards an algebraic *notation*.

The date of DIOPHANTUS can now be fixed with fair certainty. He was later than Hypsicles, from whom he quotes a definition of a polygonal number, and earlier than Theon of Alexandria, who has a quotation from Diophantus's definition of polygonal numbers. The possible limits of date are therefore, say, 150 B.C. to A.D. 250. But the letter of Psellus already mentioned says that A. M. Ptolemy (Bishop of Laodicea about A.D. 280) dedicated to Diophantus a concise treatise on the Egyptian method of reckoning, whence Diophantus must have been a contemporary, so that he probably flourished A.D. 250 or not much later.

An epigram in the Anthology gives some personal particulars of his life: his boyhood lasted $\frac{1}{6}$ th of his life; his beard grew after 5 years more; he married after $\frac{1}{7}$ th more, and his son was born 4 years later; the son lived to half his father's age, and then died 4 years after his son. Thus, if x was his age when he died,

$$\frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{2}x + 4 = x,$$

which gives $x = 84$.

Works of Diophantus.

The works on which the fame of Diophantus rests are

- (1) the *Arithmetica* (originally in thirteen Books)
- (2) a tract *On Polygonal Numbers*.

Books only of the former and a fragment of the latter
ve.

usions in the *Arithmetica* imply the existence of

A collection of propositions under the title of *Porisms*;
ree propositions (3, 5, 16) of Book V, Diophantus quotes
nown certain propositions in the Theory of Numbers,
ing to the statement of them the words 'We have it in
Porisms that...'

scholium on a passage of Iamblichus, where Iamblichus
a dictum of certain Pythagoreans about the unit being
ividing line ($\mu\epsilon\theta\acute{o}\rho\iota\omicron\nu$) between number and aliquot parts,
thus Diophantus in the *Moriastica*.... for he describes
arts" the progression without limit in the direction of
han the unit'. The *Moriastica* may be a separate work
ophantus giving rules for reckoning with fractions; but
not feel sure that the reference may not simply be to the
tions at the beginning of the *Arithmetica*.

The *Arithmetica*.

The seven lost Books and their place.

ne of the manuscripts which we possess contain more
six Books of the *Arithmetica*, the only variations being
some few divide the six Books into seven, while one or
give the fragment on Polygonal Numbers as VIII. The
ng Books were evidently lost at a very early date.
ery suggests that Hypatia's commentary extended only
e first six Books, and that she left untouched the remain-
even, which, partly as a consequence, were first forgotten
hen lost (cf. the case of Apollonius's *Conics*, where the
Books which have survived in Greek, I-IV, are those
hich Eutocius commented). There is no sign that even
rabians ever possessed the missing Books. The *Fakhrī*,
gebraical treatise by Abū Bekr Muḥ. b. al-Ḥasan al-
57 (1166) (1220) contains a list of the Books of the

and III, are 25 problems not found in Diophantus' internal evidence, and especially the admission of irrational results (which are always avoided by Diophantus), the hypothesis that we have here one of the lost Books. Nor is there any sign that more of the work than was known to Abū'l Wafā al-Būzjānī (A.D. 940-98) was a 'commentary on the algebra of Diophantus', as a 'Book of proofs of propositions used by Diophantus in his work'. These facts again point to the conclusion that the lost Books were lost before the tenth century.

The old view of the place originally occupied by the seven Books is that of Nesselmann, who argued it with great ability.¹ According to him (1) much less of Diophantus' work was wanting than would naturally be supposed on the basis of the numerical proportion of 7 lost to 6 extant Books. The missing portion came, not at the end, but in the middle of the work, and indeed mostly between the first and second Books. Nesselmann's general argument is that, if we fully read the last four Books, from the third to the sixth, we shall find that Diophantus moves in a rigidly defined and limited circle of methods and artifices, and seems in fact to be at the end of his resources. As regards the possible contents of the lost portion on this hypothesis, Nesselmann points to (1) topics which we should expect to find in the lost portion either because foreshadowed by the author himself, or necessary for the elucidation or completion of the work on the subject, (2) the *Porisms*; under head (1) come, (a) indeterminate equations of the second degree, and (b) indeterminate equations of the first degree. Diophantus does indeed show how to solve the general quadratic $ax^2 \pm bx \pm c$, so far as it has rational and positive solutions; the suitability of this would have been between Books I and II. There is nothing whatever to show that indeterminate equations of the first degree formed part of the writer's plan. Nesselmann is far from accounting for the contents of the whole Books; and he is forced to the conjecture that the seven Books may originally have been divided into even more than seven Books; there is, however, no evidence to support

¹ Nesselmann, *Algebra der Griechen*, pp. 264-73.

Relation of the 'Porisms' to the Arithmetica.

Did the *Porisms* form part of the *Arithmetica* in its original form? The phrase in which they are alluded to, and which occurs three times, 'We have it in the *Porisms* that...' suggests that they were a distinct collection of propositions concerning the properties of certain numbers, their divisibility into a certain number of squares, and so on; and it is possible that it was from the same collection that Diophantus took the numerous other propositions which he assumes, explicitly or implicitly. If the collection was part of the *Arithmetica*, it would be strange to quote the propositions under a separate title 'The Porisms' when it would have been more natural to refer to particular propositions of particular Books, and more natural still to say *τοῦτο γὰρ προδέδεικται*, or some such phrase, 'for this has been proved', without any reference to the particular place where the proof occurred. The expression 'We have it in the *Porisms*' (in the plural) would be still more inappropriate if the *Porisms* had been, as Tannery supposed, not collected together as one or more Books of the *Arithmetica*, but scattered about in the work as *corollaries* to particular propositions. Hence I agree with the view of Hultsch that the *Porisms* were not included in the *Arithmetica* at all, but formed a separate work.

If this is right, we cannot any longer hold to the view of Nesselmann that the lost Books were in the middle and not at the end of the treatise; indeed Tannery produces strong arguments in favour of the contrary view, that it is the last and most difficult Books which are lost. He replies first to the assumption that Diophantus could not have proceeded to problems more difficult than those of Book V. 'If the fifth or the sixth Book of the *Arithmetica* had been lost, who, pray, among us would have believed that such problems had ever been attempted by the Greeks? It would be the greatest error, in any case in which a thing cannot clearly be proved to have been attempted by all the ancients to maintain that

not a sufficient gap to require seven Books to fill without attributing to the ancients what modern mathematicians have discovered, may not a number of problems attributed to the Indians and Arabs have been of Greek sources? May not the same be said of problems solved by Leonardo of Pisa, which is very similar to Diophantus but is not now to be found in the *Arithmetica*. In fact, it may fairly be said that, when Chasles made a reasonably probable restitution of the *Porisms* of Apollonius notwithstanding that he had Pappus's lemmas to guide him, he undertook a more difficult task than he would have done if he had attempted to fill up seven Diophantine numerical problems which the Greeks may reasonably be supposed to have solved.¹

It is not so easy to agree with Tannery's view of the treatise *On Polygonal Numbers* to the *Arithmetica*. According to him, just as Serenus's treatise on the areas of cones and cylinders was added to the mutilated *Arithmetica* of Apollonius consisting of four Books only, in order to form a convenient volume, so the tract on Polygonal Numbers was added to the remains of the *Arithmetica*, though it was not a part of the larger work.² Thus Tannery would suppose the genuineness of the whole tract on Polygonal Numbers though in his text he only signalizes the portion of it with the enunciation of the problem 'Given a number, find in how many ways it can be a polygonal number.' The attempt by a commentator to solve this problem is not on the other hand, thinks that we may conclude that Diophantus really solved the problem. The tract on Polygonal Numbers in Book I of the *Arithmetica*, with definitions and propositions; then comes the difficult problem of the discussion of which breaks off in our text after a few lines and to these it would be easy to tack on a great number of other problems.

The name of Diophantus was used, as were those of Euclid, Archimedes and Heron in their turn, for the purpose of palming off the compilations of much later

¹ Diophantus, ed. Tannery, vol. ii, p. xx.

² *Ib.*, p. xviii.

Tannery includes in his edition three fragments under the heading 'Diophantus Pseudepigraphus'. The first, which is not 'from the Arithmetic of Diophantus' as its heading states, is worth notice as containing some particulars of one of 'two methods of finding the square root of any square number'; we are told to begin by writing the number 'according to the arrangement of the Indian method', i.e. in the Indian numerical notation which reached us through the Arabs. The second fragment is the work edited by C. Henry in 1879 as *Opusculum de multiplicatione et divisione sexagesimalibus Diophanto vel Pappo attribuendum*. The third, beginning with Διοφάντου ἐπιπεδομετρικά is a Byzantine compilation from later reproductions of the γεωμετρούμενα and στερεομετρούμενα of Heron. Not one of the three fragments has anything to do with Diophantus.

Commentators from Hypatia downwards.

The first commentator on Diophantus of whom we hear is Hypatia, the daughter of Theon of Alexandria; she was murdered by Christian fanatics in A.D. 415. I have already mentioned the attractive hypothesis of Tannery that Hypatia's commentary extended only to our six Books, and that this accounts for their survival when the rest were lost. It is possible that the remarks of Psellus (eleventh century) at the beginning of his letter about Diophantus, Anatolius and the Egyptian method of arithmetical reckoning were taken from Hypatia's commentary.

Georgius Pachymeres (1240 to about 1310) wrote in Greek a paraphrase of at least a portion of Diophantus. Sections 25-44 of this commentary relating to Book I, Def. 1 to Prop. 11, survive. Maximus Planudes (about 1260-1310) also wrote a systematic commentary on Books I, II. Arabian commentators were Abū'l Wafā al-Būzjānī (940-98), Qusṭā b. Lūqā al-Ba'labakkī (d. about 912) and probably Ibn al-Haitham (about 965-1039).

introduction to a course of lectures on astronomy gave at Padua in 1463-4 he observed: 'No translated from the Greek into Latin the fine theory of Diophantus, in which the very flower of theoretical arithmetic lies hid, the *ars rei et census* which we call by the Arabic name of Algebra.' Again, in a letter of February 5, 1464, to Bianchini, he writes that he had seen in Venice 'Diofantus, a Greek arithmetician whose work has been translated into Latin'. Rafael Bombelli was the first to find a manuscript in the Vatican and to conceive the idea of publishing the work; this was towards 1570. In 1572 Antonio Maria Pazzi, he translated five Books of the work into seven into which the manuscript was divided. The work was not published, but Bombelli took all the problems of the first four Books and some of those of the fifth and arranged them in his *Algebra* (1572), interspersing them with his own problems.

The next writer on Diophantus was Wilhelm Xylander who called himself Xylander, and who with extraordinary industry and care produced a very meritorious Latin translation with commentary (1575). Xylander was a great admirer of Diophantus, and his preface and notes are often of great value for reading. Unfortunately the book is now very scarce. The standard edition of Diophantus till recent years was that of Bachet, who in 1621 published for the first time a French text with Latin translation and notes. A second edition (1670) was carelessly printed and is untrustworthy in many places; the text; on the other hand it contained the excellent notes of Fermat; the editor was S. Fermat, his name is a great blot on the work of Bachet is his attitude towards the work to whose translation he owed more than he was willing to avow. Unfortunately neither Bachet nor Xylander used the best manuscripts; that used by Bachet was a copy of 2379 (of the middle of the sixteenth century), was made from a transcription of part of a Vatican MS. (Vatican MS. of the sixteenth century), while Xylander's manuscript was the Wolfenbüttel MS. Guelferbytanus Gudianus 1 (fifteenth century). The best and most ancient manuscript is that of Madrid (Matritensis 48 of the thirteenth century).

unfortunately spoiled by corrections made, especially in Books from some manuscript of the 'Planudean' class; where is the case recourse must be had to Vat. gr. 191 which was copied from it before it had suffered the general alteration referred to: these are the first two of the manuscripts used by Heiberg in his definitive edition of the Greek text (Teubner, 1895).

Other editors can only be shortly enumerated. In 1585 Simon Stevin published a French version of the first four books, based on Xylander. Albert Girard added the fifth and sixth Books, the complete edition appearing in 1625. German translations were brought out by Otto Schulz in 1822 and by Vertheim in 1890. Poselger translated the fragment on Polygonal Numbers in 1810. All these translations depended on the text of Bachet.

The reproduction of Diophantus in modern notation with introduction and notes by the present writer (second edition 1909) is based on the text of Tannery and may claim to be the most complete and up-to-date edition.

Any account of the *Arithmetica* of Diophantus will be most conveniently arranged under three main headings (1) the notation and definitions, (2) the principal methods employed, or as they can be generally stated, (3) the nature of the problems, including the assumed Porisms, with indications of the devices by which the problems are solved.

Notation and definitions.

In his work *Die Algebra der Griechen* Nesselmann distinguishes three stages in the evolution of algebra. (1) The first stage he calls 'Rhetorical Algebra' or reckoning by means of complete words. The characteristic of this stage is the absolute want of all symbols, the whole of the calculation being carried on by means of complete words and forming a continuous process. This first stage is represented by

literary style, but marked by the use of certain abbrevi-
 symbols for constantly recurring quantities and op
 To this stage belong Diophantus and, after him, all t
 Europeans until about the middle of the seventeenth
 (with the exception of Vieta, who was the first to e
 under the name of *Logistica speciosa*, as distinct from *L*
numerosa, a regular system of reckoning with letters o
 magnitudes as well as numbers). (3) To the thi
 Nesselmann gives the name of 'Symbolic Algebra'
 uses a complete system of notation by signs having n
 connexion with the words or things which they re
 a complete language of symbols, which entirely suppl
 'rhetorical' system, it being possible to work out a
 without using a single word of ordinary language v
 exception of a connecting word or two here and there
 clearness' sake.

Sign for the unknown ($= x$), and its origin

Diophantus's system of notation then is merely a
 tional. We will consider first the representation
 unknown quantity (our x). Diophantus defines the u
 quantity as '*containing an indeterminate or undefine*
tude of units' (πλήθος μονάδων ἀόριστον), adding t
 called ἀριθμός, i.e. *number* simply, and is denoted by
 sign. This sign is then used all through the book.
 earliest (the Madrid) MS. the sign takes the form
 Marcianus 308 it appears as S. In the printed ed
 Diophantus before Tannery's it was represented by
 sigma with an accent, σ' , which is sufficiently like th
 of the two forms. Where the symbol takes the
 inflected forms ἀριθμόν, ἀριθμοῦ, &c., the termination
 above and to the right of the sign like an exponent, e
 ἀριθμόν as τ'' for τὸν, σ'' for ἀριθμοῦ; the symbol
 addition, doubled in the plural cases, thus $\sigma\sigma'$ $\sigma\sigma\sigma'$

that the sign was not duplicated for the plural, although such duplication was the practice of the Byzantines. That the sign was merely an abbreviation for the word $\acute{\alpha}\rho\iota\theta\mu\acute{o}\varsigma$ and no algebraical symbol is shown by the fact that it occurs in the manuscripts for $\acute{\alpha}\rho\iota\theta\mu\acute{o}\varsigma$ in the ordinary sense as well as for $\acute{\alpha}\rho\iota\theta\mu\acute{o}\varsigma$ in the technical sense of the unknown quantity. Nor is it confined to Diophantus. It appears in more or less similar forms in the manuscripts of other Greek mathematicians, e.g. in the Bodleian MS. of Euclid (D'Orville 301) of the ninth century (in the forms ξ $\xi\xi$, or as a curved line similar to the abbreviation for $\kappa\alpha\iota$), in the manuscripts of the *Sand-reckoner* of Archimedes (in a form approximating to s), where again there is confusion caused by the similarity of the signs for $\acute{\alpha}\rho\iota\theta\mu\acute{o}\varsigma$ and $\kappa\alpha\iota$, in a manuscript of the *Geodaesia* included in the Heronian collections edited by Hultsch (where it appears in various forms resembling sometimes ξ , sometimes ρ , sometimes \circ , and once ξ , with case-endings superposed) and in a manuscript of Theon of Smyrna.

What is the origin of the sign? It is certainly not the final sigma, as is proved by several of the forms which it takes. I found that in the Bodleian manuscript of Diophantus it is written in the form $\zeta\xi$, larger than and quite unlike the final sigma. This form, combined with the fact that in one place Xylander's manuscript read $\alpha\rho$ for the full word, suggested to me that the sign might be a simple contraction of the first two letters of $\acute{\alpha}\rho\iota\theta\mu\acute{o}\varsigma$. This seemed to be confirmed by Gardthausen's mention of a contraction for $\alpha\rho$, in the form $\sigma\rho$ occurring in a papyrus of A.D. 154, since the transition to the form found in the manuscripts of Diophantus might easily have been made through an intermediate form ς . The loss of the downward stroke, or of the loop, would give a close approximation to the forms which we know. This hypothesis as to the origin of the sign has not, so far as I know, been improved upon. It has the immense advantage that it makes the sign for $\acute{\alpha}\rho\iota\theta\mu\acute{o}\varsigma$ similar to the signs for the powers of

The powers of the unknown, corresponding to our x^2 , are defined and denoted as follows:

x^2 is δύναμις and is denoted by Δ^Y ,

x^3 „ κύβος „ „ „ K^Y ,

x^4 „ δυναμοδύναμις „ „ $\Delta^Y \Delta$,

x^5 „ δυναμόκύβος „ „ ΔK^Y ,

x^6 „ κυβόκύβος „ „ „ $K^Y K$.

Beyond the sixth power Diophantus does not go. It be noted that, while the terms from κύβος onwards used for the powers of any ordinary known number as for the powers of the unknown, δύναμις is restricted square of the unknown; wherever a particular square is spoken of, the term is τετράγωνος ἀριθμός. The δυναμοδύναμις occurs once in another author, namely *Metrica* of Heron,¹ where it is used for the fourth p the side of a triangle.

Diophantus has also terms and signs for the reciprocals of the various powers of the unknown, i.e. for $1/x$, $1/x^2$, etc. As an aliquot part was ordinarily denoted by the corresponding numeral sign with an accent, e.g. $\gamma' = \frac{1}{3}$, $\iota\alpha' = \frac{1}{11}$, Diophantus has a mark appended to the symbols for x , x^2 ... to denote reciprocals; this, which is used for aliquot parts as printed by Tannery thus, χ . With Diophantus then

ἀριθμοστόν, denoted by ς^X , is equivalent to $1/x$

δυναμοστόν, „ $\Delta^Y X$ „ „ $1/x^2$

and so on.

The coefficient of the term in x , x^2 ... or $1/x$, $1/x^2$... expressed by the ordinary numeral immediately follows, e.g. $\Delta K^Y \kappa\varsigma = 26x^5$, $\Delta^Y X \sigma\nu = 250/x^2$.

Diophantus does not need any signs for the operations of multiplication and division. Addition is indicated by juxtaposition; thus $K^Y \alpha \Delta^Y \iota\gamma \varsigma \epsilon$ corresponds to $x^3 + 13x^2 + 250x + 1$.

¹ Heron, *Metrica*, p. 48. 11, 19, Schöne.

the abbreviation M ; thus $K^x \propto \Delta^x \gamma s \in M \beta$ corresponds to $x^3 + 13x^2 + 5x + 2$.

The sign (Λ) for minus and its meaning.

For subtraction alone is a sign used. The full term for *wanting* is $\lambda\epsilon\acute{\iota}\psi\iota\varsigma$, as opposed to $\acute{\upsilon}\pi\alpha\rho\acute{\xi}\iota\varsigma$, a *forthcoming*, which denotes a *positive* term. The symbol used to indicate a *wanting*, corresponding to our sign for *minus*, is Λ , which is described in the text as a ' ψ turned downwards and truncated' ($\Psi \acute{\epsilon}\lambda\lambda\iota\pi\acute{\epsilon}\varsigma \kappa\acute{\alpha}\tau\omega \nu\epsilon\delta\omicron\nu$). The description is evidently interpolated, and it is now certain that the sign has nothing to do with ψ . Nor is it confined to Diophantus, for it appears in practically the same form in Heron's *Metrica*,¹ where in one place the reading of the manuscript is $\mu\omicron\nu\acute{\alpha}\delta\omega\nu \omicron\delta \top \iota'\delta'$, $74 - \frac{1}{14}$. In the manuscripts of Diophantus, when the sign is resolved by writing the full word instead of it, it is generally resolved into $\lambda\epsilon\acute{\iota}\psi\epsilon\iota$, the dative of $\lambda\epsilon\acute{\iota}\psi\iota\varsigma$, but in other places the symbol is used instead of parts of the verb $\lambda\epsilon\acute{\iota}\pi\epsilon\iota\nu$, namely $\lambda\iota\pi\acute{\omega}\nu$ or $\lambda\epsilon\acute{\iota}\psi\alpha\varsigma$ and once even $\lambda\acute{\iota}\pi\omega\sigma\iota$; sometimes $\lambda\epsilon\acute{\iota}\psi\epsilon\iota$ in the manuscripts is followed by the *accusative*, which shows that in these cases the sign was wrongly resolved. It is therefore a question whether Diophantus himself ever used the dative $\lambda\epsilon\acute{\iota}\psi\epsilon\iota$ for *minus* at all. The use is certainly foreign to classical Greek. Ptolemy has in two places $\lambda\epsilon\acute{\iota}\psi\alpha\nu$ and $\lambda\epsilon\acute{\iota}\pi\omicron\nu\sigma\alpha\nu$ respectively followed, properly, by the *accusative*, and in one case he has $\tau\omicron\delta \acute{\alpha}\pi\omicron\delta \tau\eta\varsigma \Gamma\Lambda \lambda\epsilon\iota\phi\theta\acute{\epsilon}\nu \acute{\upsilon}\pi\omicron\delta \tau\omicron\upsilon \acute{\alpha}\pi\omicron\delta \tau\eta\varsigma \text{Z}\Gamma$ (where the meaning is $\text{Z}\Gamma^2 - \Gamma\Lambda^2$). Hence Heron would probably have written a participle where the \top occurs in the expression quoted above, say $\mu\omicron\nu\acute{\alpha}\delta\omega\nu \omicron\delta \lambda\epsilon\iota\psi\alpha\sigma\acute{\omega}\nu \tau\epsilon\sigma\sigma\alpha\rho\alpha\kappa\alpha\iota\delta\acute{\epsilon}\kappa\alpha\tau\omicron\nu$. On the whole, therefore, it is probable that in Diophantus, and wherever else it occurred, Λ is a compendium for the root of the verb $\lambda\epsilon\acute{\iota}\pi\epsilon\iota\nu$, in fact a Λ with ι placed in the middle (cf. λ , an abbreviation for $\tau\acute{\alpha}\lambda\alpha\nu\tau\omicron\nu$). This is the hypothesis which I put forward in 1885, and it seems to be confirmed by the fresh evidence now available as shown above.

¹ Heron, *Metrica*, p. 156. 8, 10.

Attached to the definition of *minus* is the statement that 'a *wanting* (i.e. a *minus*) multiplied by a *wanting* makes a *forthcoming* (i.e. a *plus*); and a *wanting* (a *minus*) multiplied by a *forthcoming* (a *plus*) makes a *wanting* (a *minus*)'.

Since Diophantus uses no sign for *plus*, he has to put all the positive terms in an expression together and write all the negative terms together after the sign for *minus*; e.g. for $x^3 - 5x^2 + 8x - 1$ he necessarily writes $K^Y \alpha \varsigma \eta \Lambda \Delta^Y \epsilon \overset{\circ}{M} \alpha$.

The Diophantine notation for fractions as well as for large numbers has been fully explained with many illustrations in Chapter II above. It is only necessary to add here that, when the numerator and denominator consist of composite expressions in terms of the unknown and its powers, he puts the numerator first followed by $\epsilon\nu \mu\omicron\rho\acute{\iota}\omega$ or $\mu\omicron\rho\acute{\iota}\omega$ and the denominator.

$$\begin{aligned} \text{Thus} \quad \Delta^Y \xi \overset{\circ}{M} \beta \phi \kappa \epsilon\nu \mu\omicron\rho\acute{\iota}\omega \Delta^Y \Delta \alpha \overset{\circ}{M} \gamma \Lambda \Delta^Y \xi \\ = (60x^2 + 2520)/(x^4 + 900 - 60x^2), \quad [\text{VI. 12}] \end{aligned}$$

$$\begin{aligned} \text{and} \quad \Delta^Y \iota \epsilon \Lambda \overset{\circ}{M} \lambda \varsigma \epsilon\nu \mu\omicron\rho\acute{\iota}\omega \Delta^Y \Delta \alpha \overset{\circ}{M} \lambda \varsigma \Lambda \Delta^Y \iota \beta \\ = (15x^2 - 36)/(x^4 + 36 - 12x^2) \quad [\text{VI. 14}] \end{aligned}$$

For a *term* in an algebraical expression, i.e. a power of x with a certain coefficient, and the term containing a certain number of units, Diophantus uses the word $\epsilon\acute{\iota}\delta\omicron\varsigma$, 'species' which primarily means the particular power of the variable without the coefficient. At the end of the definitions he gives directions for simplifying equations until each side contains positive terms only, by the addition or subtraction of coefficients, and by getting rid of the negative terms (which is done by adding the necessary quantities to both sides); the object, he says, is to reduce the equation until one term only is left on each side; 'but', he adds, 'I will show you later how, in the case also where two terms are left equal to one term

three terms is clearly assumed in several places of the *Arithmetica*, but Diophantus never gives the necessary explanation of this case as promised in the preface.

Before leaving the notation of Diophantus, we may observe that the form of it limits him to the use of one unknown at a time. The disadvantage is obvious. For example, where we can begin with any number of unknown quantities and gradually eliminate all but one, Diophantus has practically to perform his eliminations beforehand so as to express every quantity occurring in the problem in terms of only one unknown. When he handles problems which are by nature indeterminate and would lead in our notation to an indeterminate equation containing two or three unknowns, he has to assume for one or other of these some particular number arbitrarily chosen, the effect being to make the problem determinate. However, in doing so, Diophantus is careful to say that we may for such and such a quantity put any number whatever, say such and such a number; there is therefore (as a rule) no real loss of generality. The particular numbers by which he contrives to express all his unknowns in terms of one unknown are extraordinarily various and numerous. He can, of course, use the same variable s in the same problem with different significations *successively*, as it is necessary in the course of the problem to solve subsidiary problems in order to enable him to make the coefficients of the different terms of expressions in x such as will answer his purpose and enable the original problem to be solved. There are, however, two cases, II. 28, 29, where the proper working-out of the problem two unknowns are relatively necessary. We should of course use x and y ; but Diophantus calls the first s as usual; the second, for want of a better term, he agrees to call in the first instance 'one unit',

Then later, having completed the part of the solution necessary to find x , he substitutes its value and uses s over again for what he had originally called 1. That is, he has to

It should be premised that Diophantus will have solutions no numbers whatever except 'rational' numbers; he admits fractional solutions as well as integral, but excludes not only surds and imaginary quantities but also negative quantities. Of a negative quantity *per se*, i.e. without some greater positive quantity to subtract it from, he had apparently no conception. Such equations then leading to imaginary or negative roots he regards as useless for any purpose; the solution is in these cases *ἀδύνατος*, impossible. So we find him (V. 2) describing the equation $4 = 4x + x^2$ as *ἄτροπος*, absurd, because it would give $x = -4$. He does not, however, make occasional use of a quadratic which would give a root which is positive but a surd, but only for the purpose of obtaining limits to the root which are integers or numbers or fractions; he never uses or tries to express the actual value of such an equation. When therefore he arrives in the course of solution at an equation which would give an 'irrational' result, he retraces his steps, finds out how his equation has arisen, and how he may, by altering the previous equation, substitute for it another which shall give a rational result. This gives rise in general to a subsidiary problem the solution of which ensures a rational result for the problem itself.

It is difficult to give a complete account of Diophantus's methods without setting out the whole book, so great a variety of devices and artifices employed in the solution of the problems. There are, however, a few general methods which do admit of differentiation and description, and these we proceed to set out under subjects.

I. Diophantus's treatment of equations.

(A) *Determinate equations.*

Diophantus solved without difficulty determinate equations of the first and second degrees; of a cubic we find only one example in the *Arithmetica*, and that is a very special case.

(1) *Pure determinate equations.*

Diophantus gives a general rule for this case without exception to the first or second degree. We have to take like from like on both sides.

ion and neutralize negative terms by adding to both then take like from like again, until we have one term equal to one term. After these operations have been formed, the equation (after dividing out, if both sides in a power of x , by the lesser power) reduces to $Ax^m = B$, is considered solved. Diophantus regards this as giving root only, excluding any negative value as 'impossible'. An equation of the kind is admitted which does not give 'rational' value, integral or fractional. The value $x = 0$ is excluded in the case where the degree of the equation is reduced by dividing out by any power of x .

Fixed quadratic equations.

Diophantus never gives the explanation of the method of solution which he promises in the preface. That he had some definite method like that used in the Geometry of Heron is suggested by clear verbal explanations in different propositions. He requires the equation to be in the form of two positive terms being equal to one positive term, the possible forms for which Diophantus are

$$(a) mx^2 + px = q, \quad (b) mx^2 = px + q, \quad (c) mx^2 + q = px.$$

It does not appear that Diophantus divided by m in order to make the first term a square; rather he multiplied by m for convenience. It is clear that he stated the roots in the above form in a form equivalent to

$$(a) \frac{-\frac{1}{2}p + \sqrt{(\frac{1}{4}p^2 + mq)}}{m}, \quad (b) \frac{\frac{1}{2}p + \sqrt{(\frac{1}{4}p^2 + mq)}}{m},$$

$$(c) \frac{\frac{1}{2}p + \sqrt{(\frac{1}{4}p^2 - mq)}}{m}.$$

The explanations which show this are to be found in VI. 6, VII. 39 and 31, and in V. 10 and VI. 22 respectively. For example in V. 10 he has the equation $17x^2 + 17 < 72x$, and he says 'Multiply half the coefficient of x into itself and we have 152, subtract the product of the coefficient of x^2 and the

equation $2x^2 > 6x + 18$ and says, 'To solve this, take the square of half the coefficient of x , i.e. 9, and the product of the unit-term and the coefficient of x^2 , i.e. 36. Adding, we have 45, the square root of which is not less than 7. Add half the coefficient of x [and divide by the coefficient of x^2]; whence x is not less than 5.' In these cases it will be observed that 31 and 7 are not accurate limits, but are the nearest integral limits which will serve his purpose.

Diophantus always uses the positive sign with the radical, and there has been much discussion as to whether he knew that a quadratic equation has *two* roots. The evidence of the text is inconclusive because his only object, in every case, is to get one solution; in some cases the other root would be negative, and would therefore naturally be ignored as 'absurd' or 'impossible'. In yet other cases where the second root is possible it can be shown to be useless from Diophantus's point of view. For my part, I find it difficult or impossible to believe that Diophantus was unaware of the existence of two real roots in such cases. It is so obvious from the geometrical form of solution based on Eucl. II. 5, 6 and that contained in Eucl. VI. 27-9; the construction of VI. 28, too, corresponds in fact to the *negative* sign before the radical in the case of the particular equation there solved, while a quite obvious and slight variation of the construction would give the solution corresponding to the *positive* sign.

The following particular cases of quadratics occurring in the *Arithmetica* may be quoted, with the results stated by Diophantus.

$$x^2 = 4x - 4; \text{ therefore } x = 2. \quad (\text{IV. 22})$$

$$325x^2 = 3x + 18; x = \frac{78}{325} \text{ or } \frac{6}{25}. \quad (\text{IV. 31})$$

$$84x^2 + 7x = 7; x = \frac{1}{4}. \quad (\text{VI. 6})$$

$$84x^2 - 7x = 7; x = \frac{1}{3}. \quad (\text{VI. 7})$$

$$630x^2 - 73x = 6; x = \frac{6}{35}. \quad (\text{VI. 9})$$

$$630x^2 + 73x = 6; x \text{ is rational.} \quad (\text{VI. 8})$$

the first and third of the last three cases the limits are not rate, but are *integral* limits which are *a fortiori* safe. The second $\frac{6}{1} \frac{6}{9}$ should have been $\frac{6}{1} \frac{7}{9}$, and it would have been correct to say that, if x is not greater than $\frac{6}{1} \frac{7}{9}$ and not than $\frac{6}{1} \frac{7}{9}$, the given conditions are *a fortiori* satisfied. For comparison with Diophantus's solutions of quadratic equations we may refer to a few of his solutions of

Simultaneous equations involving quadratics.

In I. 27, 28, and 30 we have the following pairs of equations.

$$\left. \begin{array}{l} \xi + \eta = 2a \\ \xi\eta = B \end{array} \right\}, \quad (\beta) \quad \left. \begin{array}{l} \xi + \eta = 2a \\ \xi^2 + \eta^2 = B \end{array} \right\}, \quad (\gamma) \quad \left. \begin{array}{l} \xi - \eta = 2a \\ \xi\eta = B \end{array} \right\}.$$

Use the Greek letters for the numbers required to be found distinct from the one unknown which Diophantus uses, and which I shall call x .

(α), he says, let $\xi - \eta = 2x$ ($\xi > \eta$).

It follows, by addition and subtraction, that $\xi = a + x$, $\eta = a - x$;

therefore $\xi\eta = (a + x)(a - x) = a^2 - x^2 = B$,

x is found from the pure quadratic equation.

(β) similarly he assumes $\xi - \eta = 2x$, and the resulting

equation is $\xi^2 + \eta^2 = (a + x)^2 + (a - x)^2 = 2(a^2 + x^2) = B$.

(γ) he puts $\xi + \eta = 2x$ and solves as in the case of (α).

Cubic equation.

Only one very particular case occurs. In VI. 17 the problem is reduced to the equation

$$x^2 + 2x + 3 = x^3 + 3x - 3x^2 - 1.$$

Diophantus says simply 'whence x is found to be 4'. In fact the equation reduces to

Diophantus says nothing of indeterminate equations of first degree. The reason is perhaps that it is a principle with him to admit rational *fractional* as well as integral solutions, whereas the whole point of indeterminate equations of first degree is to obtain a solution in *integral* numbers. Without this limitation (foreign to Diophantus) such equations have no significance.

(a) *Indeterminate equations of the second degree.*

The form in which these equations occur is invariably one or two (but never more) functions of x of the form $Ax^2 + Bx + C$ or simpler forms are to be made rational numbers by finding a suitable value for x . That is, to solve, in the most general case, one or two equations of the form $Ax^2 + Bx + C = y^2$.

(1) *Single equation.*

The solutions take different forms according to the particular values of the coefficients. Special cases arise when one or more of them vanish or they satisfy certain conditions.

1. When A or C or both vanish, the equation can be solved rationally.

Form $Bx = y^2$.

Form $Bx + C = y^2$.

Diophantus puts for y^2 any determinate square m^2 , and x is immediately found.

Form $Ax^2 + Bx = y^2$.

Diophantus puts for y any multiple of x , as $\frac{m}{n}x$.

2. The equation $Ax^2 + C = y^2$ can be rationally solved according to Diophantus:

(a) when A is positive and a square, say a^2 ;
in this case we put $a^2x^2 + C = (ax \pm m)^2$, whence

$$x = \pm \frac{C - m^2}{2ma}$$

(m and the sign being so chosen as to give x a positive value).

when C is positive and a square, say c^2 ;
 s case Diophantus puts $Ax^2 + c^2 = (mx \pm c)^2$, and obtains

$$x = \pm \frac{2mc}{A - m^2}.$$

When one solution is known, any number of other
 ons can be found. This is stated in the Lemma to
 5. It would be true not only of the cases $\pm Ax^2 \mp C = y^2$,
 of the general case $Ax^2 + Bx + C = y^2$. Diophantus, how-
 only states it of the case $Ax^2 - C = y^2$.

s method of finding other (greater) values of x satisfy-
 the equation when one (x_0) is known is as follows. If
 $-C = q^2$, he substitutes in the original equation $(x_0 + x)$
 and $(q - kx)$ for y , where k is some integer.

en, since $A(x_0 + x)^2 - C = (q - kx)^2$, while $Ax_0^2 - C = q^2$,
 ows by subtraction that

$$2x(Ax_0 + kq) = x^2(k^2 - A),$$

$$x = 2(Ax_0 + kq) / (k^2 - A),$$

$$\text{the new value of } x \text{ is } x_0 + \frac{2(Ax_0 + kq)}{k^2 - A}.$$

$$\text{m } Ax^2 - c^2 = y^2.$$

phantus says (VI. 14) that a rational solution of this
 s only possible when A is the sum of two squares.

fact, if $x = p/q$ satisfies the equation, and $Ax^2 - c^2 = k^2$,

$$\text{ve } Ap^2 = c^2q^2 + k^2q^2,$$

$$A = \left(\frac{cq}{p}\right)^2 + \left(\frac{kq}{p}\right)^2.]$$

$$\text{m } Ax^2 + C = y^2.$$

phantus proves in the Lemma to VI. 12 that this equa-
 as an infinite number of solutions when $A + C$ is a square,
 the particular case where $x = 1$ is a solution. (He does

substituting in the original equation $1+x$ for x and $(q-ka)$ for y , where k is some integer.

3. Form $Ax^2+Bx+C=y^2$.

This can be reduced to the form in which the second term is wanting by replacing x by $z - \frac{B}{2A}$.

Diophantus, however, treats this case separately and less fully. According to him, a rational solution of the equation $Ax^2+Bx+C=y^2$ is only possible

(α) when A is positive and a square, say a^2 ;

(β) when C is positive and a square, say c^2 ;

(γ) when $\frac{1}{4}B^2-AC$ is positive and a square.

In case (α) y is put equal to $(ax-m)$, and in case (β) y is put equal to $(mx-c)$.

Case (γ) is not expressly enunciated, but occurs, as it were, accidentally (IV. 31). The equation to be solved is $3x+18-x^2=y^2$. Diophantus first assumes $3x+18-x^2=4x^2$ which gives the quadratic $3x+18=5x^2$; but this 'is not rational'. Therefore the assumption of $4x^2$ for y^2 will not do 'and we must find a square [to replace 4] such that 18 times (this square+1)+ $(\frac{3}{2})^2$ may be a square'. 'The auxiliary equation is therefore $18(m^2+1)+\frac{9}{4}=y^2$, or $72m^2+81=y^2$, a square, and Diophantus assumes $72m^2+81=(8m+9)^2$, whence $m=18$. Then, assuming $3x+18-x^2=(18)^2x^2$, he obtains the equation $325x^2-3x-18=0$, whence $x=\frac{78}{325}$, that is, $\frac{6}{25}$.

(2) *Double equation.*

The Greek term is $\delta\iota\pi\lambda\omicron\iota\sigma\acute{o}\tau\eta\varsigma$, $\delta\iota\pi\lambda\hat{\eta}$ $\iota\sigma\acute{o}\tau\eta\varsigma$ or $\delta\iota\pi\lambda\hat{\eta}$ $\iota\sigma\omega\sigma\iota\varsigma$. Two different functions of the unknown have to be made simultaneously squares. The general case is to solve in rational numbers the equations

$$\left. \begin{aligned} mx^2+\alpha x+a &= u^2 \\ nx^2+\beta x+b &= w^2 \end{aligned} \right\}.$$

able equation of the first degree.

the equations are

$$\alpha x + a = u^2,$$

$$\beta x + b = w^2.$$

ophantus has one general method taking slightly different
s according to the nature of the coefficients.

First method of solution.

is depends upon the identity

$$\left\{\frac{1}{2}(p+q)\right\}^2 - \left\{\frac{1}{2}(p-q)\right\}^2 = pq.$$

the difference between the two expressions in x can be
ated into two factors p, q , the expressions themselves
equated to $\left\{\frac{1}{2}(p+q)\right\}^2$ and $\left\{\frac{1}{2}(p-q)\right\}^2$ respectively. As
nantus himself says in II. 11, we 'equate either the square
lf the difference of the two factors to the lesser of the
essions, or the square of half the sum to the greater'.

e will consider the general case and investigate to what
cular classes of cases the method is applicable from
nantus's point of view, remembering that the final quad-
in x must always reduce to a single equation.

stracting, we have $(\alpha - \beta)x + (a - b) = u^2 - w^2$.

parate $(\alpha - \beta)x + (a - b)$ into the factors

$$p, \{(\alpha - \beta)x + (a - b)\} / p.$$

e write accordingly

$$u \pm w = \frac{(\alpha - \beta)x + (a - b)}{p},$$

$$u \mp w = p.$$

us $u^2 = \alpha x + a = \frac{1}{4} \left\{ \frac{(\alpha - \beta)x + (a - b)}{p} + p \right\}^2$;

fore $\{(\alpha - \beta)x + a - b + p^2\}^2 = 4p^2(\alpha x + a).$

In order that this equation may reduce to a simple equation either

(1) the coefficient of x^2 must vanish, or $\alpha - \beta = 0$,

or (2) the absolute term must vanish, that is,

$$p^4 - 2p^2(a+b) + (a-b)^2 = 0,$$

or $\{p^2 - (a+b)\}^2 = 4ab$,

so that ab must be a square number.

As regards condition (1) we observe that it is really sufficient if $\alpha n^2 = \beta m^2$, since, if $\alpha x + a$ is a square, $(\alpha x + a)n^2$ is equal to a square, and, if $\beta x + b$ is a square, so is $(\beta x + b)m^2$, and vice versa.

That is, (1) we can solve any pair of equations of the form

$$\left. \begin{aligned} \alpha m^2 x + a &= u^2 \\ \alpha n^2 x + b &= w^2 \end{aligned} \right\}.$$

Multiply by n^2 , m^2 respectively, and we have to solve the equations

$$\left. \begin{aligned} \alpha m^2 n^2 x + a n^2 &= u'^2 \\ \alpha m^2 n^2 x + b m^2 &= w'^2 \end{aligned} \right\}.$$

Separate the difference, $an^2 - bm^2$, into two factors p , q and put

$$u' \pm w' = p,$$

$$u' \mp w' = q;$$

therefore $u'^2 = \frac{1}{4}(p+q)^2$, $w'^2 = \frac{1}{4}(p-q)^2$,

and $\alpha m^2 n^2 x + a n^2 = \frac{1}{4}(p+q)^2$,

$$\alpha m^2 n^2 x + b m^2 = \frac{1}{4}(p-q)^2;$$

and from either of these equations we get

$$x = \frac{\frac{1}{4}(p^2 + q^2) - \frac{1}{2}(an^2 + bm^2)}{\alpha m^2 n^2},$$

Ex. from Diophantus:

$$\left. \begin{aligned} 65 - 6x &= u^2 \\ 65 - 24x &= w^2 \end{aligned} \right\}; \quad (\text{IV. } 32)$$

therefore

$$\left. \begin{aligned} 260 - 24x &= u'^2 \\ 65 - 24x &= w'^2 \end{aligned} \right\}.$$

The difference $= 195 = 15 \cdot 13$, say;

therefore $\frac{1}{4}(15 - 13)^2 = 65 - 24x$; that is, $24x = 64$, and $x = \frac{8}{3}$.

Taking now the condition (2) that ab is a square, we see that the equations can be solved in the cases where either a and b are both squares, or the ratio of a to b is the ratio of a square to a square. If the equations are

$$\alpha x + c^2 = u^2,$$

$$\beta x + d^2 = w^2,$$

and factors are taken of the difference between the expressions as they stand, then, since one factor p , as we saw, satisfies the equation

$$\{p^2 - (c^2 + d^2)\}^2 = 4c^2d^2,$$

we must have

$$p = c \pm d.$$

Ex. from Diophantus:

$$\left. \begin{aligned} 10x + 9 &= u^2 \\ 5x + 4 &= w^2 \end{aligned} \right\}. \quad (\text{III. } 15)$$

The difference is $5x + 5 = 5(x + 1)$; the solution is given by $(\frac{1}{2}x + 3)^2 = 10x + 9$, and $x = 28$.

Another method is to multiply the equations by squares such that, when the expressions are subtracted, the absolute term vanishes. The case can be worked out generally, thus.

Multiply by d^2 and c^2 respectively, and we have to solve

$$\left. \begin{aligned} \alpha d^2 x + c^2 d^2 &= u^2 \\ \beta c^2 x + c^2 d^2 &= w^2 \end{aligned} \right\}.$$

Difference $= (\alpha d^2 - \beta c^2)x = \dots$, say

or, since

$$pq = \alpha d^2 - \beta c^2,$$

$$p^2 x^2 - 2x(\alpha d^2 + \beta c^2) + q^2 - 4c^2 d^2 = 0.$$

In order that this may reduce to a simple equation, Diophantus requires, the absolute term must vanish, so that $q = 2cd$. The method therefore only gives one solution, as q is restricted to the value $2cd$.

Ex. from Diophantus:

$$\left. \begin{aligned} 8x + 4 &= u^2 \\ 6x + 4 &= w^2 \end{aligned} \right\}.$$

Difference $2x$; q necessarily taken to be $2\sqrt{4}$ or 4; therefore $\frac{1}{2}x, 4$. Therefore $8x + 4 = \frac{1}{4}(\frac{1}{2}x + 4)^2$, and $x = 4$.

(β) Second method of solution of a double equation of the first degree.

There is only one case of this in Diophantus, the equation being of the form

$$\left. \begin{aligned} hx + n^2 &= u^2 \\ (h + f)x + n^2 &= w^2 \end{aligned} \right\}.$$

Suppose $hx + n^2 = (y + n)^2$; therefore $hx = y^2 + 2ny$

and $(h + f)x + n^2 = (y + n)^2 + \frac{f}{h}(y^2 + 2ny)$.

It only remains to make the latter expression a square, which is done by equating it to $(py - n)^2$.

The case in Diophantus is the same as that last mentioned (IV. 39). Where I have used y , Diophantus as usual chooses to use his one unknown a second time.

2. Double equations of the second degree.

The general form is

$$Ax^2 + Bx + C = u^2$$

$$(2) \quad \left. \begin{aligned} x^2 + \alpha x + a &= u^2 \\ \beta x^2 + a &= w^2 \end{aligned} \right\}.$$

(The case where the absolute terms are in the ratio of a square to a square reduces to this.)

In all examples of these cases the usual method of solution applies.

$$(3) \quad \left. \begin{aligned} \alpha x^2 + ax &= u^2 \\ \beta x^2 + bx &= w^2 \end{aligned} \right\}.$$

The usual method does not here serve, and a special artifice is required.

Diophantus assumes $u^2 = m^2 x^2$.

Then $x = a/(m^2 - \alpha)$ and, by substitution in the second equation, we have

$$\beta \left(\frac{a}{m^2 - \alpha} \right)^2 + \frac{ba}{m^2 - \alpha}, \text{ which must be made a square,}$$

or $a^2 \beta + ba(m^2 - \alpha)$ must be a square.

We have therefore to solve the equation

$$abm^2 + a(a\beta - \alpha b) = y^2,$$

which can or cannot be solved by Diophantus's methods according to the nature of the coefficients. Thus it can be solved if $(a\beta - \alpha b)/a$ is a square, or if a/b is a square. Examples in VI. 12, 14.

(b) *Indeterminate equations of a degree higher than the second.*

(1) *Single equations.*

There are two classes, namely those in which expressions in x have to be made squares or cubes respectively. The general form is therefore

The species of the first class found in the *Arithmetica* as follows.

1. Equation $Ax^3 + Bx^2 + Cx + d^2 = y^2$.

As the absolute term is a square, we can assume

$$y = \frac{C}{2d}x + d,$$

or we might assume $y = m^2x^2 + nx + d$ and determine m that the coefficients of x , x^2 in the resulting equation vanish.

Diophantus has only one case, $x^3 - 3x^2 + 3x + 1 = y^2$ (V) and uses the first method.

2. Equation $Ax^4 + Bx^3 + Cx^2 + Dx + E = y^2$, where either E is a square.

If A is a square ($= a^2$), we may assume $y = ax^2 + \frac{B}{2a}$ determining n so that the term in x^2 in the resulting equation may vanish. If E is a square ($= e^2$), we may assume $y = mx^2 + \frac{D}{2e}x + e$, determining m so that the term in x^2 in the resulting equation may vanish. We shall then, in either case, obtain a simple equation in x .

3. Equation $Ax^4 + Cx^2 + E = y^2$, but in special cases only all the coefficients are squares.
4. Equation $Ax^4 + E = y^2$.

The case occurring in Diophantus is $x^4 + 97 = y^2$ (I). Diophantus tries one assumption, $y = x^2 - 10$, and finds this gives $x^2 = \frac{3}{25}$, which leads to no rational result; he therefore goes back and alters his assumptions so that he is able to replace the refractory equation by $x^4 + 33 = y^2$, and at the same time to find a suitable value for y , namely $y = x^2 - 25$, which produces a rational result, $x = \frac{1}{5}$.

5. Equation of sixth degree in the special form

are. Where this does not hold (in IV. 18) Diophantus goes back and replaces the equation $x^6 - 16x^3 + x + 64 = y^2$ by another, $x^6 - 128x^3 + x + 4096 = y^2$.

expressions which have to be made *cubes*, we have the following cases.

$$x^2 + Bx + C = y^3.$$

There are only two cases of this. First, in VI. 1, $x^2 - 4x + 4$ is to be made a cube, being already a square. Diophantus actually makes $x - 2$ a cube.

Secondly, a peculiar case occurs in VI. 17, where a cube has to be found exceeding a square by 2. Diophantus assumes $(x+1)^3$ for the cube and $(x+1)^2$ for the square. This gives

$$x^3 - 3x^2 + 3x - 1 = x^2 + 2x + 3,$$

or $x^3 + x = 4x^2 + 4$. We divide out by $x^2 + 1$, and $x = 4$. It is evident that the assumptions were made with knowledge and intention. That is, Diophantus knew of the solution 27 and 25 and deliberately led up to it. It is unlikely that he was aware of the fact, observed by Fermat, that 27 and 25 are the only integral numbers satisfying the condition.

$Ax^3 + Bx^2 + Cx + D = y^3$, where either A or D is a cube number, or both are cube numbers. Where A is a cube (a^3),

we have only to assume $y = ax + \frac{B}{3a^2}$, and where D is a cube

$y = \frac{C}{3a^2}x + d$. Where $A = a^3$ and $D = d^3$, we can use

either assumption, or put $y = ax + d$. Apparently Diophantus used the last assumption only in this case, for in IV. 27 he treats as impossible the equation $8x^3 - x^2 + 8x - 1 = y^3$, while the assumption $y = 2x - 1$ gives a negative value $-\frac{1}{11}$, whereas either of the above assumptions gives a rational value.

c) *Double equations.*

Where one expression has to be made a square and another

$$\left. \begin{aligned} 2x^2 + 2x &= y^2 \\ x^3 + 2x^2 + x &= z^3 \end{aligned} \right\}.$$

Diophantus assumes $y = mx$, whence $x = 2/(m^2 - 2)$

$$\left(\frac{2}{m^2 - 2}\right)^3 + 2\left(\frac{2}{m^2 - 2}\right)^2 + \frac{2}{m^2 - 2} = z^3,$$

or
$$\frac{2m^4}{(m^2 - 2)^3} = z^3.$$

We have only to make $2m^4$, or $2m$, a cube.

II. Method of Limits.

As Diophantus often has to find a series of order of magnitude, and as he does not admit solutions, it is often necessary for him to reject found in the usual course because it does not necessary conditions; he is then obliged, in many find solutions lying *within certain limits* in place rejected. For example:

1. It is required to find a value of x such that some it, x^n , shall lie between two given numbers, say a

Diophantus multiplies both a and b by 2^n , 3^n successively, until some n th power is seen which the two products. Suppose that c^n lies between a then we can put $x = c/p$, for $(c/p)^n$ lies between

Ex. To find a square between $1\frac{1}{4}$ and 2. multiplies by a square 64; this gives 80 and 1 which lies 100. Therefore $(\frac{10}{8})^2$ or $\frac{25}{16}$ solves (IV. 31 (2)).

To find a sixth power between 8 and 16. The squares of 1, 2, 3, 4 are 1, 64, 729, 4096. Multiply 8 and 16 by 64 and we have 512 and 1024, between which 729 is therefore a solution (VI. 21).

2. Sometimes a value of x has to be found which

no other functions of x .

Ex. 1. In IV. 25 a value of x is required such that $8/(x^2+x)$ lie between x and $x+1$.

The part of the condition gives $8 > x^3+x^2$. Diophantus accordingly assumes $8 = (x + \frac{1}{3})^3 = x^3 + x^2 + \frac{1}{3}x + \frac{1}{27}$, which is $x^2 + x^2$. Thus $x + \frac{1}{3} = 2$ or $x = \frac{5}{3}$ satisfies one part of the condition. Incidentally it satisfies the other, namely $(x^2+x) < x+1$. This is a piece of luck, and Diophantus is satisfied with it, saying nothing more.

Ex. 2. We have seen how Diophantus concludes that, if

$$\frac{1}{5}(x^2-60) > x > \frac{1}{8}(x^2-60),$$

x is not less than 11 and not greater than 12 (V. 30).

The problem further requires that x^2-60 shall be a square. Assuming $x^2-60 = (x-m)^2$, we find $x = (m^2+60)/2m$. Since $x > 11$ and < 12 , says Diophantus, it follows that

$$24m > m^2+60 > 22m;$$

which he concludes that m lies between 19 and 21. Taking $m = 20$, he finds $x = 11\frac{1}{2}$.

III. Method of approximation to Limits.

Here we have a very distinctive method called by Diophantus *ρότης* or *παρισότητος ἀγωγή*. The object is to solve such problems as that of finding two or three square numbers the sum of which is a given number, while each of them either approximates to one and the same number, or is subject to conditions which may be the same or different. Two examples will best show the method.

Ex. 1. Divide 13 into two squares each of which > 6 (V. 9). Take half of 13, i.e. $6\frac{1}{2}$, and find what *small* fraction $1/x^2$ added to it will give a square;

$$6\frac{1}{2} + \frac{1}{x^2}, \text{ or } 26 + \frac{1}{y^2}, \text{ must be a square.}$$

Diophantus assumes

$$26 + \frac{1}{y^2} = \left(5 + \frac{1}{y}\right)^2, \text{ or } 26y^2 + 1 = (5y + 1)^2,$$

whence

$$y = 10, \text{ and } 1/y^2 = \frac{1}{100}, \text{ i.e. } 1/x^2 = \frac{1}{400}; \text{ and } 6\frac{1}{2} + \frac{1}{400} = \left(\frac{51}{20}\right)^2$$

[The assumption of $5 + \frac{1}{y}$ as the side is not haphazard: 5 is chosen because it is the most suitable as giving the largest rational value for y .]

We have now, says Diophantus, to divide 13 into two squares each of which is as nearly as possible equal to $\left(\frac{51}{20}\right)^2$.

Now $13 = 3^2 + 2^2$ [it is necessary that the original number shall be capable of being expressed as the sum of two squares]

and

$$3 > \frac{51}{20} \text{ by } \frac{9}{20},$$

while

$$2 < \frac{51}{20} \text{ by } \frac{11}{20}.$$

But if we took $3 - \frac{9}{20}$, $2 + \frac{11}{20}$ as the sides of two squares their sum would be $2\left(\frac{51}{20}\right)^2 = \frac{5202}{400}$, which is > 13 .

Accordingly we assume $3 - 9x$, $2 + 11x$ as the sides of the required squares (so that x is not exactly $\frac{1}{20}$ but near it).

Thus

$$(3 - 9x)^2 + (2 + 11x)^2 = 13,$$

and we find $x = \frac{5}{101}$.

The sides of the required squares are $\frac{257}{101}$, $\frac{258}{101}$.

Ex. 2. Divide 10 into three squares each of which $>$ (V. 11).

[The original number, here 10, must of course be expressible as the sum of three squares.]

Take one-third of 10, i.e. $3\frac{1}{3}$, and find what small fraction $1/x^2$ added to it will make a square; i.e. we have to make

$3\frac{1}{3} + \frac{1}{x^2}$ a square, i.e. $30 + \frac{9}{x^2}$ must be a square, or $30 + \frac{9}{x^2} = y^2$, where $3/x = 1/y$.

fore $y = 2$, and $1/x^2 = \frac{1}{36}$; and $3\frac{1}{3} + \frac{1}{36} = \frac{121}{36}$, a square.

have now, says Diophantus, to divide 10 into three parts with sides as near as may be to $\frac{11}{6}$.

$$10 = 9 + 1 = 3^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2.$$

Adding $3, \frac{3}{5}, \frac{4}{5}$ and $\frac{11}{6}$ to a common denominator, we have $\frac{18}{6}, \frac{24}{30}$ and $\frac{55}{30}$,

$$3 > \frac{55}{30} \text{ by } \frac{35}{30},$$

$$\frac{3}{5} < \frac{55}{30} \text{ by } \frac{37}{30},$$

$$\frac{4}{5} < \frac{55}{30} \text{ by } \frac{31}{30}.$$

Now we took $3 - \frac{35}{30}, \frac{3}{5} + \frac{37}{30}, \frac{4}{5} + \frac{31}{30}$ as the sides of squares, the sum of the squares would be $3\left(\frac{11}{6}\right)^2$ or $\frac{363}{36}$, which is > 10 . Accordingly we assume as the sides $3 - 35x, \frac{3}{5} + 37x, \frac{4}{5} + 31x$, where x must therefore be not exactly $\frac{1}{30}$ but near it.

$$\text{Hence } (3 - 35x)^2 + \left(\frac{3}{5} + 37x\right)^2 + \left(\frac{4}{5} + 31x\right)^2 = 10,$$

$$10 - 116x + 3555x^2 = 10,$$

$$\text{and } x = \frac{116}{3555};$$

the sides of the required squares are $\frac{1321}{711}, \frac{1285}{711}, \frac{1288}{711}$;

the squares themselves are $\frac{1745041}{505521}, \frac{1651225}{505521}, \frac{1658944}{505521}$.

Other instances of the application of the method will be found in V. 10, 12, 13, 14.

THEOREMS AND PROPOSITIONS IN THE THEORY OF NUMBERS.

Three propositions are quoted as occurring in the *Porisms* (we have it in the *Porisms* that...'); and some other propositions assumed without proof may very likely have come from the same collection. The three propositions from the *Porisms* are to the following effect.

If a is a given number and x, y numbers such that $x^2 = m^2, y + a = n^2$, then, if $xy + a$ is also a square, m and n are both by unity (V. 3).

be taken equal to $2\{m^2 + (m+1)^2\} + 2$, or $4(m^2 + m)$. The three numbers have the property that the product of *plus* either the sum of those two or the remaining gives a square (V. 5).

[In fact, if X, Y, Z denote the numbers respectively

$$XY + X + Y = (m^2 + m + 1)^2, \quad XY + Z = (m^2 + m + 1)^2,$$

$$YZ + Y + Z = (2m^2 + 3m + 3)^2, \quad YZ + X = (2m^2 + 3m + 3)^2,$$

$$ZX + Z + X = (2m^2 + m + 2)^2, \quad ZX + Y = (2m^2 + m + 2)^2.$$

3. The difference of any two cubes is also the sum of two cubes, i.e. can be transformed into the sum of two cubes (V. 16).

[Diophantus merely states this without proving it, leaving it to the reader to find out how to make the transformation. The subject of the transformation of sums and differences of cubes was investigated by Vieta, Bachet and Fermat.]

II. Of the many other propositions assumed or implied by Diophantus which are not referred to the *Porisms*, we may distinguish two classes.

1. The first class are of two sorts; some are more or less of the nature of identical formulae, e.g. the facts that $\{\frac{1}{2}(a+b)\}^2 - ab$ and $a^2(a+1)^2 + a^2 + (a+1)^2$ are respectively squares, that $a(a^2 - a) + a + (a^2 - a)$ is a square, and that 8 times a triangular number *plus* a square, i.e. $8 \cdot \frac{1}{2}x(x+1) + 1 = (2x+1)^2$. Others are of the same kind as the first two propositions quoted from the *Porisms*, e.g.

(1) If $X = a^2x + 2a$, $Y = (a+1)^2x + 2(a+1)$ or, in other words, if $xX + 1 = (ax+1)^2$ and $xY + 1 = \{(a+1)x+1\}^2$, then $XY + 1$ is a square (IV. 20). In fact

$$XY + 1 = \{a(a+1)x + (2a+1)\}^2.$$

(2) If $X \pm a = m^2$, $Y \pm a = (m+1)^2$, and $Z = 2(X \pm a)$, then $YZ \pm a$, $ZX \pm a$, $XY \pm a$ are all squares (V. 3).

$$\begin{aligned}YZ \pm a &= \{(m+1)(2m+1) \mp 2a\}^2, \\ZX \pm a &= \{m(2m+1) \mp 2a\}^2, \\XY \pm a &= \{m(m+1) \mp a\}^2.\end{aligned}$$

If

$$m^2 + 2, \quad Y = (m+1)^2 + 2, \quad Z = 2\{m^2 + (m+1)^2 + 1\} + 2,$$

the six expressions

$$\begin{aligned}YZ - (Y+Z), \quad ZX - (Z+X), \quad XY - (X+Y), \\YZ - X, \quad ZX - Y, \quad XY - Z\end{aligned}$$

all squares (V. 6).

fact

$$(Y+Z) = (2m^2 + 3m + 3)^2, \quad YZ - X = (2m^2 + 3m + 4)^2, \quad \&c.$$

The second class is much more important, consisting of propositions in the Theory of Numbers which we find first or assumed in the *Arithmetica*. It was in explanation or extension of these that Fermat's most famous notes were written. How far Diophantus possessed scientific proofs of the theorems which he assumes must remain largely a matter of speculation.

Theorems on the composition of numbers as the sum of two squares.

Any square number can be resolved into two squares in an infinite number of ways (II. 8).

Any number which is the sum of two squares can be resolved into two other squares in any number of ways (II. 9).

It is implied throughout that the squares may be fractional as well as integral.)

If there are two whole numbers each of which is the sum of two squares, the product of the numbers can be resolved into two squares.

triangles from (a, b) and (c, d) respectively, b
phantus means, form the right-angled triangles

$$(a^2 + b^2, a^2 - b^2, 2ab) \text{ and } (c^2 + d^2, c^2 - d^2, 2cd)$$

Multiply all the sides in each triangle by the
the other; we have then two rational right-angled triangles
with the same hypotenuse $(a^2 + b^2)(c^2 + d^2)$.

Two others are furnished by the formula $(a^2 - b^2, 2ab, a^2 + b^2)$
have only to 'form two right-angled triangles' $(ad - bc, a^2 + b^2, ad + bc)$
 $ad - bc$ and from $(ac - bd, ad + bc)$ respectively.
fails if certain relations hold between a, b, c, d ,
not be such that one number of either pair vanishes
that $ad = bc$ or $ac = bd$, or such that the numbers
pair are equal to one another, for then the result is
illusory.

In the case taken by Diophantus $a^2 + b^2 = 5^2 + 4^2 = 41$,
 $c^2 + d^2 = 3^2 + 2^2 = 13$, and the four right-angled triangles

$$(65, 52, 39), (65, 60, 25), (65, 63, 16) \text{ and } (65, 72, 13)$$

On this proposition Fermat has a long and interesting note
as to the number of ways in which a prime number of the form
form $4n + 1$ and its powers can be (a) the hypotenuse of
a rational right-angled triangle, (b) the sum of two squares.
He also extends theorem (3) above: 'If a prime number
is the sum of two squares be multiplied by a number which
is also the sum of two squares the product will be the sum of
two squares in two ways; if the first number be multiplied by
the square of the second, the product will be the sum of two
squares in three ways; the product of the first number by the
cube of the second will be the sum of two squares in four ways,
and so on *ad infinitum*.'

Although the hypotenuses selected by Diophantus are
prime numbers of the form $4n + 1$, it is not clear that he
was aware that prime numbers of the form $4n + 1$ are the only
numbers arising from the multiplication of such numbers.
The only classes of numbers which are always the sum of two
squares; this was first proved by Euler.

(4) More remarkable is a condition of possibility
prefixed to V. 9, 'To divide 1 into two parts

square.' The condition is in two parts. 'There is no doubt as to the first, 'The given number must not be odd' [i.e. no number of the form $4n+3$ or $4n-1$ can be the sum of two squares]; the text of the second part is corrupt, but the words actually found in the text make it quite likely that corrections made by Hankel and Tannery give the real meaning of the original, 'nor must the double of the given number *plus* 1 be measured by any prime number which is less by 1 than a multiple of 4'. This is tolerably near the true condition stated by Fermat, 'The given number must not be odd, and the double of it increased by 1, when divided by the greatest square which measures it, must not be divisible by a prime number of the form $4n-1$.'

(β) *On numbers which are the sum of three squares.*

In V. 11 the number $3a+1$ has to be divisible into three squares. Diophantus says that a 'must not be 2 or any multiple of 8 increased by 2'. That is, '*a number of the form $24n+7$ cannot be the sum of three squares*'. As a matter of fact, the factor 3 in the 24 is irrelevant here, and Diophantus might have said that a number of the form $8n+7$ cannot be the sum of three squares. The latter condition is true, but does not include *all* the numbers which cannot be the sum of three squares. Fermat gives the conditions to which a must be subject, proving that $3a+1$ cannot be of the form $4^n(24k+7)$ or $4^n(8k+7)$, where $k=0$ or any integer.

(γ) *Composition of numbers as the sum of four squares.*

There are three problems, IV. 29, 30 and V. 14, in which it is required to divide a number into four squares. Diophantus states no necessary condition in this case, as he does when it is a question of dividing a number into *three* or *two* squares. Now *every number is either a square or the sum of two, three or four squares* (a theorem enunciated by Fermat and proved by Lagrange who followed up results obtained by Euler), and this shows that any number can be divided into four squares (admitting fractional as well as integral squares), since any square number can be divided into two other squares, integral

or fractional. It is possible, therefore, that Diophantus was *empirically* aware of the truth of the theorem of Fermat, but we cannot be sure of this.

Conspectus of the *Arithmetica*, with typical solutions

There seems to be no means of conveying an idea of the extent of the problems solved by Diophantus except by giving a conspectus of the whole of the six Books. Fortunately this can be done by the help of modern notation without occupying too many pages.

It will be best to classify the propositions according to their character rather than to give them in Diophantus's order. It should be premised that $x, y, z \dots$ indicating the first, second, and third \dots numbers required do not mean that Diophantus indicates any of them by his unknown (s); he gives his unknown known in each case the signification which is most convenient, his object being to express all his required numbers at once in terms of the one unknown (where possible), thereby avoiding the necessity for eliminations. Where I have occasion to speak of Diophantus's unknown, I shall as a rule call it ξ , except when a problem includes a subsidiary problem and it is convenient to use different letters for the unknown in the original and subsidiary problems respectively, in order to mark clearly the distinction between them. When in the equations expressions are said to be $= u^2, v^2, w^2, t^2 \dots$ this means simply that they are to be made squares. Given numbers will be indicated by $a, b, c \dots m, n \dots$ and will take the place of the number s by Diophantus, which are always specific numbers.

Where the solutions, or particular devices employed, are specially ingenious or interesting, the methods of solution will be shortly indicated. The character of the book will be better appreciated by means of such illustrations.

[The problems marked with an asterisk are probably spurious.]

$$\text{I. 9. } a - x = m(b - x).$$

$$\text{I. 10. } x + b = m(a - x).$$

$$\text{I. 11. } x + b = m(x - a).$$

$$\begin{aligned} \text{I. 39. } & \left. \begin{aligned} (a+x)b + (b+x)a &= 2(a+b)x, \\ \text{or } (a+b)x + (b+x)a &= 2(a+x)b, \\ \text{or } (a+b)x + (a+x)b &= 2(b+x)a. \end{aligned} \right\} (a > b) \end{aligned}$$

Diophantus states this problem in this form, 'Given two numbers (a, b) , to find a third number (x) such that the numbers

$$(a+x)b, (b+x)a, (a+b)x$$

are in arithmetical progression.'

The result is of course different according to the order of magnitude of the three expressions. If $a > b$ (5 and 3 are the numbers in Diophantus), then $(a+x)b < (b+x)a$; there are consequently three alternatives, since $(a+x)b$ must be either the least or the middle, and $(b+x)a$ either the middle or the greatest of the three products. We may have

$$(a+x)b < (a+b)x < (b+x)a,$$

$$\text{or } (a+b)x < (a+x)b < (b+x)a,$$

$$\text{or } (a+x)b < (b+x)a < (a+b)x,$$

and the corresponding equations are as set out above.

(ii) Determinate systems of equations of the first degree.

$$\text{I. 1. } x + y = a, \quad x - y = b.$$

$$\left\{ \begin{array}{l} \text{I. 2. } x + y = a, \quad x = my, \\ \text{I. 4. } x - y = a, \quad x = my. \end{array} \right.$$

$$\text{I. 3. } x + y = a, \quad x = my + b.$$

$$\left\{ \begin{array}{l} \text{I. 12. } x_1 + x_2 = y_1 + y_2 = a, x_1 = my_2, y_1 = nx_2 \ (x_1 > x_2, y_1 > y_2) \\ \text{I. 13. } x_1 + x_2 = y_1 + y_2 = z_1 + z_2 = a \\ \quad x_1 = my_2, y_1 = nz_2, z_1 = px_2 \end{array} \right\} \ (x_1 > x_2, y_1 > y_2, z_1 > z_2)$$

$$\text{I. 15. } x + a = m(y - a), y + b = n(x - b).$$

[Diophantus puts $y = \xi + a$, where ξ is his unknown.]

$$\left\{ \begin{array}{l} \text{I. 16. } y + z = a, z + x = b, x + y = c. \text{ [Dioph. puts } \xi = x + y + z.] \\ \text{I. 17. } y + z + w = a, z + w + x = b, w + x + y = c, x + y + z = d. \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{I. 18. } y + z - x = a, z + x - y = b, x + y - z = c. \\ \quad \text{[Dioph. puts } 2\xi = x + y + z.] \\ \text{I. 19. } y + z + w - x = a, z + w + x - y = b, w + x + y - z = c, x + y + z - w = d. \end{array} \right.$$

$$\text{I. 20. } x + y + z = a, x + y = mz, y + z = nx.$$

$$\text{I. 21. } x = y + \frac{1}{m}z, y = z + \frac{1}{n}x, z = a + \frac{1}{p}y \text{ (where } x > y > z \text{)}$$

with necessary condition.

$$\begin{aligned} \text{II. 18*} \quad x - \left(\frac{1}{m}x + a\right) + \left(\frac{1}{p}z + c\right) &= y - \left(\frac{1}{n}y + b\right) + \left(\frac{1}{m}x + a\right) \\ &= z - \left(\frac{1}{p}z + c\right) + \left(\frac{1}{n}y + b\right), x + y + z = 0 \end{aligned}$$

[Solution wanting.]

(iii) Determinate systems of equations reducible to the first degree.

$$\text{I. 26. } ax = \alpha^2, bx = \alpha.$$

$$\text{I. 29. } x + y = a, x^2 - y^2 = b. \text{ [Dioph. puts } 2\xi = x - y.]$$

$$\text{I. 31. } x = my, x^2 + y^2 = n(x + y).$$

$$\text{I. 32. } x = my, x^2 + y^2 = n(x - y).$$

I. 35. $x = my, y^2 = nx.$

I. 36. $x = my, y^2 = ny.$

I. 37. $x = my, y^2 = n(x+y).$

I. 38. $x = my, y^2 = n(x-y).$

I. 38. Cor. $x = my, x^2 = ny.$

„ $x = my, x^2 = nx.$

„ $x = my, x^2 = n(x+y).$

„ $x = my, x^2 = n(x-y).$

II. 6*. $x-y = a, x^2-y^2 = x-y+b.$

IV. 36. $yz = m(y+z), zx = n(z+x), xy = p(x+y).$

[Solved by means of Lemma: see under (vi) Indeterminate equations of the first degree.]

(iv) Determinate systems reducible to equations of second degree.

I. 27. $x+y = a, xy = b.$

[Dioph. states the necessary condition, namely that $\frac{1}{4}a^2 - b$ must be a square, with the words *ἔστι δὲ τοῦτο πλασματικόν*, which no doubt means 'this is of the nature of a formula (easily obtained)'. He puts $x-y = 2\xi.$]

I. 30. $x-y = a, xy = b.$

[Necessary condition (with the same words) $4b + a^2 =$ a square. $x+y$ is put $= 2\xi.$]

I. 28. $x+y = a, x^2+y^2 = b.$

[Necessary condition $2b - a^2 =$ a square. $x-y = 2\xi.$]

IV. 1. $x^3+y^3 = a, x+y = b.$

[Dioph. puts $x-y = 2\xi$, whence $x = \frac{1}{2}b + \xi, y = \frac{1}{2}b - \xi.$

IV. 15. $(y+z)x = a$, $(z+x)y = b$, $(x+y)z = c$.

[Dioph. takes the third number z as his unknown

thus

$$x + y = c/z.$$

Assume $x = p/z$, $y = q/z$. Then

$$\frac{pq}{z^2} + p = a,$$

$$\frac{pq}{z^2} + q = b.$$

These equations are inconsistent unless $p = q$. We have therefore to determine p, q by dividing the two parts such that their difference $= a - b$ (cf. p. 487).

A very interesting use of the 'false hypothesis' (Diophantus first takes two *arbitrary* numbers p, q such that $p + q = c$, and finds that the values of a, b to be corrected).

The final equation being $\frac{pq}{z^2} + p = a$, where p, q are determined in the way described, $z^2 = pq / (a - p)$, $pq / (b - q)$, and the numbers a, b, c have to be so chosen that either of these expressions gives a square.]

IV. 34. $yz + (y+z) = a^2 - 1$, $zx + (z+x) = b^2 - 1$,

$$xy + (x+y) = c^2 - 1,$$

[Dioph. states as the necessary condition for a solution that each of the three constants to be subtracted from the three expressions are to be equal must be so chosen that when diminished by 1. The true condition is so chosen that in notation by transforming the equations $yz + (y+z) = a^2 - 1$, $zx + (z+x) = b^2 - 1$, $xy + (x+y) = c^2 - 1$ into

whence
$$x+1 = \sqrt{\left\{ \frac{(\beta+1)(\gamma+1)}{\alpha+1} \right\}} \&c.;$$

and it is only necessary that $(\alpha+1)(\beta+1)(\gamma+1)$ should be a square, not that *each* of the expressions $\alpha+1$, $\beta+1$, $\gamma+1$ should be a square.

Dioph. finds in a Lemma (see under (vi) below) a solution $\epsilon\nu \acute{\alpha}\rho\acute{\iota}\sigma\tau\varphi$ (indeterminately) of $xy + (x+y) = k$, which practically means finding y in terms of x .]

IV. 35. $yz - (y+z) = a^2 - 1$, $zx - (z+x) = b^2 - 1$,

$$xy - (x+y) = c^2 - 1.$$

[The remarks on the last proposition apply *mutatis mutandis*. The lemma in this case is the indeterminate solution of $xy - (x+y) = k$.]

IV. 37. $yz = a(x+y+z)$, $zx = b(x+y+z)$, $xy = c(x+y+z)$.

[Another interesting case of 'false hypothesis'. Dioph. first gives $x+y+z$ an *arbitrary* value, then finds that the result is not rational, and proceeds to solve the new problem of finding a value of $x+y+z$ to take the place of the first value.

If $w = x+y+z$, we have $x = cw/y$, $z = aw/y$, so that $zx = acw^2/y^2 = bw$ by hypothesis; therefore $y^2 = \frac{ac}{b}w$.

For a rational solution this last expression must be a square. Suppose, therefore, that $w = \frac{ac}{b}\xi^2$, and we have

$$x+y+z = \frac{ac}{b}\xi^2, \quad y = \frac{ac}{b}\xi, \quad z = a\xi, \quad x = c\xi.$$

Eliminating x , y , z , we obtain $\xi = (bc + ca + ab)/ac$,
and

(v) Systems of equations apparently indeterminate really reduced, by arbitrary assumptions, to determinate equations of the first degree.

I. 14. $xy = m(x+y)$. [Value of y arbitrarily assumed.]

$$\left\{ \begin{array}{l} \text{II. 3*}. \quad xy = m(x+y), \text{ and } xy = m(x-y). \\ \text{II. 1*}. \quad (\text{cf. I. 31}). \quad x^2 + y^2 = m(x+y). \\ \text{II. 2*}. \quad (\text{cf. I. 34}). \quad x^2 - y^2 = m(x-y). \\ \text{II. 4*}. \quad (\text{cf. I. 32}). \quad x^2 + y^2 = m(x-y). \\ \text{II. 5*}. \quad (\text{cf. I. 33}). \quad x^2 - y^2 = m(x+y). \end{array} \right\} \quad [x \text{ assumed.}]$$

II. 7*. $x^2 - y^2 = m(x-y) + a$. [Dioph. assumes $x-y = a$.]

$$\left\{ \begin{array}{l} \text{I. 22.} \quad x - \frac{1}{m}x + \frac{1}{p}z = y - \frac{1}{n}y + \frac{1}{m}x = z - \frac{1}{p}z + \frac{1}{n}z \\ \quad \quad \quad [\text{Value of } y \text{ assumed.}] \\ \text{I. 23.} \quad x - \frac{1}{m}x + \frac{1}{q}w = y - \frac{1}{n}y + \frac{1}{m}x = z - \frac{1}{p}z + \frac{1}{n}z \\ \quad \quad \quad = w - \frac{1}{q}w + \frac{1}{p}z. \quad [\text{Value of } y \text{ assumed.}] \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{I. 24.} \quad x + \frac{1}{m}(y+z) = y + \frac{1}{n}(z+x) = z + \frac{1}{p}(x+y). \\ \quad \quad \quad [\text{Value of } y+z \text{ assumed.}] \\ \text{I. 25.} \quad x + \frac{1}{m}(y+z+w) = y + \frac{1}{n}(z+w+x) \\ \quad \quad \quad = z + \frac{1}{p}(w+x+y) = w + \frac{1}{q}(x+y+z). \\ \quad \quad \quad [\text{Value of } y+z+w \text{ assumed.}] \end{array} \right.$$

II. 17*. (cf. I. 22). $x - \left(\frac{1}{m}x + a\right) + \left(\frac{1}{p}z + c\right)$

$$\text{IV. 33. } x + \frac{1}{z}y = m\left(y - \frac{1}{z}y\right), \quad y + \frac{1}{z}x = n\left(x - \frac{1}{z}x\right).$$

[Dioph. assumes $\frac{1}{z}y = 1$.]

(vi) Indeterminate equations of the first degree.

Lemma to IV. 34. $xy + (x + y) = a$.
 „ „ IV. 35. $xy - (x + y) = a$.
 „ „ IV. 36. $xy = m(x + y)$.
 [Solutions $\epsilon\nu \acute{\alpha}\omicron\rho\iota\sigma\tau\varphi$.
 y practically found
 in terms of x .]

(vii) Indeterminate analysis of the second degree.

II. 8. $x^2 + y^2 = a^2$.
 [$y^2 = a^2 - x^2$ must be a square $= (mx - a)^2$, say.]
 II. 9. $x^2 + y^2 = a^2 + b^2$. [Put $x = \xi + a$, $y = m\xi - b$.]
 II. 10. $x^2 - y^2 = a$.
 [Put $x = y + m$, choosing m such that $m^2 < a$.]

II. 11. $x + a = u^2$, $x + b = v^2$.
 II. 12. $a - x = u^2$, $b - x = v^2$.
 II. 13. $x - a = u^2$, $x - b = v^2$.

[Dioph. solves II. 11 and 13, (1) by means of the 'double equation' (see p. 469 above), (2) without a double equation by putting $x = \xi^2 \pm a$ and equating $(\xi^2 \pm a) \pm b$ to $(\xi - m)^2$. In II. 12 he puts $x = a - \xi^2$.]

II. 14 = III. 21. $x + y = a$, $x + z^2 = u^2$, $y + z^2 = v^2$.

[Diophantus takes z as the unknown, and puts $u^2 = (z + m)^2$, $v^2 = (z + n)^2$. Therefore $x = 2mz + m^2$, $y = 2nz + n^2$, and z is found, by substitution in the first equation, to be $\frac{a - (m^2 + n^2)}{2(m + n)}$. In order that the solution

may be rational, m, n must satisfy a certain condition. Dioph. takes them such that $m^2 + n^2 < a$, but it is sufficient if $m > n$ that a may be $> n^2$.]

II. 16. $x = my$, $u^2 + x = v^2$, $u^2 + y = v^2$.

II. 19. $x^2 - y^2 = m(y^2 - z^2)$.

II. 20. $x^2 + y = u^2$, $y^2 + x = v^2$.
 [Assume $y = 2mx + m^2$, and one condition is satisfied.]

II. 21. $x^2 - y = u^2$, $y^2 - x = v^2$.
 [Assume $x = \xi + m$, $y = 2m\xi + m^2$, and one condition is satisfied.]

II. 22. $x^2 + (x + y) = u^2$, $y^2 + (x + y) = v^2$.
 [Put $x + y = 2mx + m^2$.]

II. 23. $x^2 - (x + y) = u^2$, $y^2 - (x + y) = v^2$.

II. 24. $(x + y)^2 + x = u^2$, $(x + y)^2 + y = v^2$.
 [Assume $x = (m^2 - 1)\xi^2$, $y = (n^2 - 1)\xi^2$, $x + y = \xi$.]

II. 25. $(x + y)^2 - x = u^2$, $(x + y)^2 - y = v^2$.

II. 26. $xy + x = u^2$, $xy + y = v^2$, $u + v = a$.
 [Put $y = m^2x - 1$.]

II. 27. $xy - x = u^2$, $xy - y = v^2$, $u + v = a$.

II. 28. $x^2y^2 + x^2 = u^2$, $x^2y^2 + y^2 = v^2$.

II. 29. $x^2y^2 - x^2 = u^2$, $x^2y^2 - y^2 = v^2$.

II. 30. $xy + (x + y) = u^2$, $xy - (x + y) = v^2$.

[Since $m^2 + n^2 \pm 2mn$ is a square, assume

$$xy = (m^2 + n^2)\xi^2 \text{ and } x + y = 2mn\xi^2;$$

put $x = p\xi$, $y = q\xi$, where $pq = m^2 + n^2$; then

$$(p + q)\xi = 2mn\xi^2.]$$

II. 31. $xy + (x + y) = u^2$, $xy - (x + y) = v^2$, $x + y = w^2$.

[Suppose $w^2 = 2 \cdot 2m \cdot m$, which is a square, and use formula $(2m)^2 + m^2 + 2 \cdot 2m \cdot m = \text{a square}$]

$$\text{II. 34. } x^2 + (x + y + z) = u^2, \quad y^2 + (x + y + z) = v^2,$$

$$z^2 + (x + y + z) = w^2.$$

[Since $\{\frac{1}{2}(m-n)\}^2 + mn$ is a square, take any number separable into two factors (m, n) in three ways. This gives three values, say, p, q, r for $\frac{1}{2}(m-n)$. Put $x = p\xi, y = q\xi, z = r\xi$, and $x + y + z = mn\xi^2$; therefore $(p+q+r)\xi = mn\xi^2$, and ξ is found.]

$$\text{II. 35. } x^2 - (x + y + z) = u^2, \quad y^2 - (x + y + z) = v^2,$$

$$z^2 - (x + y + z) = w^2.$$

[Use the formula $\{\frac{1}{2}(m+n)\}^2 - mn = \text{a square}$ and proceed similarly.]

$$\text{III. 1*} \quad (x + y + z) - x^2 = u^2, \quad (x + y + z) - y^2 = v^2,$$

$$(x + y + z) - z^2 = w^2.$$

$$\text{III. 2*} \quad (x + y + z)^2 + x = u^2, \quad (x + y + z)^2 + y = v^2,$$

$$(x + y + z)^2 + z = w^2.$$

$$\text{III. 3*} \quad (x + y + z)^2 - x = u^2, \quad (x + y + z)^2 - y = v^2,$$

$$(x + y + z)^2 - z = w^2.$$

$$\text{III. 4*} \quad x - (x + y + z)^2 = u^2, \quad y - (x + y + z)^2 = v^2,$$

$$z - (x + y + z)^2 = w^2.$$

$$\text{III. 5. } x + y + z = t^2, \quad y + z - x = u^2, \quad z + x - y = v^2,$$

$$x + y - z = w^2.$$

[The first solution of this problem assumes

$$t^2 = x + y + z = (\xi + 1)^2, \quad w^2 = 1, \quad u^2 = \xi^2,$$

whence x, y, z are found in terms of ξ , and $z + x - y$ is then made a square.

The alternative solution, however, is much more elegant, and can be generalized thus.

We have to find x, y, z so that

$$\left. \begin{aligned} -x + y + z &= \text{a square} \\ x - y + z &= \text{a square} \end{aligned} \right\}$$

Then, since the sum of the first three express itself equal to $x+y+z$, we have

$$x = \frac{1}{2}(b^2 + c^2), \quad y = \frac{1}{2}(c^2 + a^2), \quad z = \frac{1}{2}(a^2 + b^2).$$

$$\text{III. 6. } x+y+z = t^2, \quad y+z = u^2, \quad z+x = v^2, \quad x+y = w^2.$$

$$\text{III. 7. } x-y = y-z, \quad y+z = u^2, \quad z+x = v^2, \quad x+y = w^2.$$

$$\left\{ \begin{array}{l} \text{III. 8. } x+y+z+a = t^2, \quad y+z+a = u^2, \quad z+x+a = v^2, \quad x+y+a = w^2. \\ \text{III. 9. } x+y+z-a = t^2, \quad y+z-a = u^2, \quad z+x-a = v^2, \quad x+y-a = w^2. \end{array} \right.$$

$$\text{III. 10. } yz+a = u^2, \quad zx+a = v^2, \quad xy+a = w^2.$$

[Suppose $yz+a = m^2$, and let $y = (m^2-a)\xi$, z also let $zx+a = n^2$; therefore $x = (n^2-a)\xi$.

We have therefore to make

$$(m^2-a)(n^2-a)\xi^2 + a \text{ a square.}$$

Diophantus takes $m^2 = 25$, $a = 12$, $n^2 = 1$ arrives at $52\xi^2 + 12$, which is to be made a square. Although $52 \cdot 1^2 + 12$ is a square, and it follows that a number of other solutions giving a square are possible by substituting $1+\eta$ for ξ in the expression, and Diophantus says that the equation could easily be solved if 52 was a square, and proceeds to solve the problem by finding two squares such that each increased by 12 give a square, in which case their product also gives a square. In other words, we have to find m^2 and n^2 such that m^2-a , n^2-a are both squares, which, as he says, is easy. We have to find two pairs of squares differing by a . If

$$a = pq = p'q', \quad \left\{ \frac{1}{2}(p-q) \right\}^2 + a = \left\{ \frac{1}{2}(p+q) \right\}^2$$

$$\text{and} \quad \left\{ \frac{1}{2}(p'-q') \right\}^2 + a = \left\{ \frac{1}{2}(p'+q') \right\}^2;$$

$$\text{let, then, } m^2 = \left\{ \frac{1}{2}(p+q) \right\}^2, \quad n^2 = \left\{ \frac{1}{2}(p'+q') \right\}^2.]$$

III. 15. $yz + (y + z) = u^2$, $zx + (z + x) = v^2$, $xy + (x + y) = w^2$.

[*Lemma.* If a , $a + 1$ be two consecutive numbers, $a^2(a + 1)^2 + a^2 + (a + 1)^2$ is a square. Let

$$y = m^2, z = (m + 1)^2.$$

Therefore $(m^2 + 2m + 2)x + (m + 1)^2$
and $(m^2 + 1)x + m^2$ }

have to be made squares. This is solved as a double-equation; in Diophantus's problem $m = 2$.

Second solution. Let x be the first number, m the second; then $(m + 1)x + m$ is a square $= n^2$, say; therefore $x = (n^2 - m)/(m + 1)$, while $y = m$. We have then

and $\left. \begin{array}{l} (m + 1)z + m = \text{a square} \\ \left(\frac{n^2 + 1}{m + 1}\right)z + \frac{n^2 - m}{m + 1} = \text{a square} \end{array} \right\}$

Diophantus has $m = 3$, $n = 5$, so that the expressions to be made squares are with him

$$\left. \begin{array}{l} 4z + 3 \\ 6\frac{1}{2}z + 5\frac{1}{2} \end{array} \right\}.$$

This is not possible because, of the corresponding coefficients, neither pair are in the ratio of squares. In order to substitute, for $6\frac{1}{2}$, 4, coefficients which are in the ratio of a square to a square he then finds two numbers, say, p , q to replace $5\frac{1}{2}$, 3 such that $pq + p + q = \text{a square}$, and $(p + 1)/(q + 1) = \text{a square}$. He assumes ξ and $4\xi + 3$, which satisfies the second condition, and then solves for ξ , which must satisfy

$$4\xi^2 + 8\xi + 3 = \text{a square} = (2\xi - 3)^2, \text{ say,}$$

which gives $\xi = \frac{3}{10}$, $4\xi + 3 = 4\frac{1}{5}$.

He then solves, for z , the third number, the double-equation

$$\left. \begin{array}{l} 5\frac{1}{5}z + 4\frac{1}{5} = \text{square} \\ \frac{1}{10}z + \frac{3}{10} = \text{square} \end{array} \right\},$$

after multiplying by 25 and 100 respectively, expressions

$$\left. \begin{array}{l} 130x + 105 \\ 130x + 30 \end{array} \right\}.$$

In the above equations we should only have $n^2 + 1$ a square, and then multiply the first by the second by $(m + 1)^2$.

Diophantus, with his notation, was hardly in to solve, as we should, by writing

$$\begin{aligned} (y + 1)(z + 1) &= a^2 + 1, \\ (z + 1)(x + 1) &= b^2 + 1, \\ (x + 1)(y + 1) &= c^2 + 1, \end{aligned}$$

which gives $x + 1 = \sqrt{\{(b^2 + 1)(c^2 + 1)/(a^2 + 1)\}}$

$$\text{III. 16. } yz - (y + z) = u^2, \quad zx - (z + x) = v^2, \quad xy - (x + y) = w^2$$

[The method is the same *mutatis mutandis* second of the above solutions.]

$$\text{III. 17. } xy + (x + y) = u^2, \quad xy + x = v^2, \quad xy + y = w^2$$

$$\text{III. 18. } xy - (x + y) = u^2, \quad xy - x = v^2, \quad xy - y = w^2$$

$$\text{III. 19. } (x_1 + x_2 + x_3 + x_4)^2 \pm x_1 = \begin{cases} t^2 \\ t'^2 \end{cases}$$

$$(x_1 + x_2 + x_3 + x_4)^2 \pm x_2 = \begin{cases} u^2 \\ u'^2 \end{cases}$$

$$(x_1 + x_2 + x_3 + x_4)^2 \pm x_3 = \begin{cases} v^2 \\ v'^2 \end{cases}$$

$$(x_1 + x_2 + x_3 + x_4)^2 \pm x_4 = \begin{cases} w^2 \\ w'^2 \end{cases}.$$

[Diophantus finds, in the way we have seen, four different rational right-angled triangles with same hypotenuse, namely (25, 52, 53), (25, 52, 53), (25, 52, 53), (25, 52, 53).

Put therefore $x_1 + x_2 + x_3 + x_4 = 65\xi$.

and $x_1 = 2.39.52\xi^2$, $x_2 = 2.25.60\xi^2$, $x_3 = 2.33.56\xi^2$,

$x_4 = 2.16.63\xi^2$;

this gives $12768\xi^2 = 65\xi$, and $\xi = \frac{65}{12768}$.]

{ IV. 4. $x^2 + y = u^2$, $x + y = u$.

{ IV. 5. $x^2 + y = u$, $x + y = u^2$.

IV. 13. $x + 1 = t^2$, $y + 1 = u^2$, $x + y + 1 = v^2$, $y - x + 1 = w^2$.

[Put $x = (m\xi + 1)^2 - 1 = m^2\xi^2 + 2m\xi$; the second and third conditions require us to find two squares with x as difference. The difference $m^2\xi^2 + 2m\xi$ is separated into the factors $m^2\xi + 2m$, ξ ; the square of half the difference $= \{\frac{1}{2}(m^2 - 1)\xi + m\}^2$. Put this equal to $y + 1$, so that $y = \frac{1}{4}(m^2 - 1)^2\xi^2 + m(m^2 - 1)\xi + m^2 - 1$, and the first three conditions are satisfied. The fourth gives $\frac{1}{4}(m^4 - 6m^2 + 1)\xi^2 + (m^3 - 3m)\xi + m^2 =$ a square, which we can equate to $(n\xi - m)^2$.]

IV. 14. $x^2 + y^2 + z^2 = (x^2 - y^2) + (y^2 - z^2) + (x^2 - z^2)$. ($x > y > z$)

IV. 16. $x + y + z = t^2$, $x^2 + y = u^2$, $y^2 + z = v^2$, $z^2 + x = w^2$.

[Put $4m\xi$ for y , and by means of the factors $2m\xi$, 2 we can satisfy the second condition by making x equal to half the difference, or $m\xi - 1$. The third condition is satisfied by subtracting $(4m\xi)^2$ from some square, say $(4m\xi + 1)^2$; therefore $z = 8m\xi + 1$. By the first condition $13m\xi$ must be a square. Let it be $169\eta^2$; the numbers are therefore $13\eta^2 - 1$, $52\eta^2$, $104\eta^2 + 1$, and the last condition gives $10816\eta^4 + 221\eta^2 =$ a square, i.e. $10816\eta^2 + 221 =$ a square $= (104\eta + 1)^2$, say. This gives the value of η , and solves the problem.]

IV. 17. $x + y + z = t^2$, $x^2 - y = u^2$, $y^2 - z = v^2$, $z^2 - x = w^2$.

IV. 19. $yz + 1 = u^2$, $zx + 1 = v^2$, $xy + 1 = w^2$.

Similarly we satisfy the second condition by as
 $zx = n^2\xi^2 + 2n\xi$; therefore $x = n^2\xi + 2n$. To sat
 third condition, we must have

$$(m^2n^2\xi^2 + 2mn \cdot \overline{m+n}\xi + 4mn) + 1 \text{ a square}$$

We must therefore have $4mn + 1$ a square a
 $mn(m+n) = mn\sqrt{(4mn+1)}$. The first cond
 satisfied by $n = m + 1$, which incidentally satis
 second condition also. We put therefore $yz = (m\xi$
 and $zx = \{(m+1)\xi + 1\}^2 - 1$, and assume that $z = \xi$
 $y = m^2\xi + 2m$, $x = (m+1)^2\xi + 2(m+1)$, and w
 shown that the third condition is also satisfied. T
 have a solution in terms of the undetermined unk
 The above is only slightly generalized from Dioph

$$\text{IV. 20. } x_2x_3 + 1 = r^2, \quad x_3x_1 + 1 = s^2, \quad x_1x_2 + 1 = t^2,$$

$$x_1x_4 + 1 = u^2, \quad x_2x_4 + 1 = v^2, \quad x_3x_4 + 1 = w^2.$$

[This proposition depends on the last, x_1, x_2, x_3 , a
 determined as in that proposition. If x_3 correspo
 in that proposition, we satisfy the condition $x_3x_4 + 1 =$
 by putting $x_3x_4 = \{(m+2)\xi + 1\}^2 - 1$, and so fin
 terms of ξ , after which we have only two conditio
 to satisfy. The condition $x_1x_4 + 1 =$ square
 matically satisfied, since

$$\{(m+1)^2\xi + 2(m+1)\} \{(m+2)^2\xi + 2(m+2)\}$$

is a square, and it only remains to satisfy $x_2x_4 + 1 =$
 That is,

$$\begin{aligned} & (m^2\xi + 2m) \{(m+2)^2\xi + 2(m+2)\} + 1 \\ & = m^2(m+2)^2\xi^2 + 2m(m+2)(2m+2)\xi + 4m(m+2) \end{aligned}$$

has to be made a square, which is easy, since the c
 of ξ^2 is a square.

With Diophantus $m = 1$, so that $x_1 = 4\xi + 4$, $x_2 = \xi$, $x_3 = 9\xi + 6$, and $9\xi^2 + 24\xi + 13$ has to
 a square. He equates this to $(3\xi - 4)^2$, giving $\xi =$

[Since $x^2 + x + \frac{1}{4}$ is a square,

$$(x^2 + x) + (y^2 + y) + (z^2 + z) + (w^2 + w) + 1$$

is the sum of four squares, and we only have to separate $a + 1$ into four squares.]

IV. 30. $x^2 + y^2 + z^2 + w^2 - (x + y + z + w) = a.$

IV. 31. $x + y = 1, (x + a)(y + b) = u^2.$

IV. 32. $x + y + z = a, xy + z = u^2, xy - z = v^2.$

IV. 39. $x - y = m(y - z), y + z = u^2, z + x = v^2, x + y = w^2.$

IV. 40. $x^2 - y^2 = m(y - z), y + z = u^2, z + x = v^2, x + y = w^2.$

{ V. 1. $xz = y^2, x - a = u^2, y - a = v^2, z - a = w^2.$

{ V. 2. $xz = y^2, x + a = u^2, y + a = v^2, z + a = w^2.$

{ V. 3. $x + a = r^2, y + a = s^2, z + a = t^2,$
 $yz + a = u^2, zx + a = v^2, xy + a = w^2.$

{ V. 4. $x - a = r^2, y - a = s^2, z - a = t^2,$
 $yz - a = u^2, zx - a = v^2, xy - a = w^2.$

[Solved by means of the *Porisms* that, if a be the given number, the numbers $m^2 - a, (m + 1)^2 - a$ satisfy the conditions of V. 3, and the numbers $m^2 + a, (m + 1)^2 + a$ the conditions of V. 4 (see p. 479 above). The third number is taken to be $2\{m^2 \mp a + (m + 1)^2 \mp a\} - 1$, and the three numbers automatically satisfy two more conditions (see p. 480 above). It only remains to make

$$2\{m^2 \mp a + (m + 1)^2 \mp a\} - 1 \pm a \text{ a square,}$$

or $4m^2 + 4m \mp 3a + 1 = \text{a square,}$

which is easily solved.

With Diophantus $\xi + 3$ takes the place of m in V. 3 and ξ takes its place in V. 4, while a is 5 in V. 3 and 6 in V. 4.]

V. 5. $y^2 z^2 + x^2 = r^2, z^2 x^2 + y^2 = s^2, x^2 y^2 + z^2 = t^2,$

$$y^2 z^2 + y^2 + z^2 = u^2, z^2 x^2 + z^2 + x^2 = v^2, x^2 y^2 + x^2 + y^2 = w^2$$

[Solved by means of the *Porism* numbered 2 on p. 480.

$$V. 6. x-2 = r^2, y-2 = s^2, z-2 = t^2,$$

$$yz-y-z = u^2, zx-z-x = v^2, xy-x-y = w^2,$$

$$yz-x = u'^2, zx-y = v'^2, xy-z = w'^2.$$

[Solved by means of the proposition numbered (3) on p. 481.]

$$\text{Lemma 1 to V. 7. } xy+x^2+y^2 = u^2.$$

$$V. 7. x^2 \pm (x+y+z) = \begin{cases} u^2 \\ u'^2 \end{cases}, y^2 \pm (x+y+z) = \begin{cases} v^2 \\ v'^2 \end{cases},$$

$$z^2 \pm (x+y+z) = \begin{cases} w^2 \\ w'^2 \end{cases}.$$

[Solved by means of the subsidiary problem (Lemma 2) of finding three rational right-angled triangles with equal area. If m, n satisfy the condition in Lemma 1, i.e. if $mn+m^2+n^2=p^2$, the triangles are 'formed' from the pairs of numbers (p, m) , (p, n) , $(p, m+n)$ respectively. Diophantus assumes this, but it is easy to prove. In his case $m=3$, $n=5$, so that $p=7$. Now, in a right-angled triangle, (hypotenuse)² \pm four times area is a square. We equate, therefore, $x+y+z$ to four times the common area multiplied by ξ^2 , and the several numbers x, y, z to the three hypotenuses multiplied by ξ , and equate the two values. In Diophantus's case the triangles are (40, 42, 58), (24, 70, 74) and (15, 112, 113), and $245\xi = 3360\xi^2$.]

$$V. 8. yz \pm (x+y+z) = \begin{cases} u^2 \\ u'^2 \end{cases}, zx \pm (x+y+z) = \begin{cases} v^2 \\ v'^2 \end{cases},$$

$$xy \pm (x+y+z) = \begin{cases} w^2 \\ w'^2 \end{cases}$$

[Solved by means of the same three rational right-angled triangles found in the Lemma to V. 7, together with the Lemma that we can solve the equations $yz=a^2$, $zx=b^2$, $xy=c^2$.]

$$\left\{ \begin{array}{l} V. 9. (\text{Cf. II. 11}). \quad x+y=1, x+a=u^2, y+a=v^2. \\ V. 11. \quad x+y+z=1, x+a=u^2, y+a=v^2, z+a=w^2. \end{array} \right.$$

[These are the problems of $\pi\alpha\rho\iota\sigma\delta\tau\eta\tau\omicron\varsigma \delta\gamma\omega\gamma\eta$

described above (pp. 477-9). The problem is 'to divide unity into two (or three) parts such that, if one and the same given number be added to each part, the results are all squares'.]

$$10. x+y=1, x+a=u^2, y+b=v^2.$$

$$12. x+y+z=1, x+a=u^2, y+b=v^2, z+c=w^2.$$

[These problems are like the preceding except that *different* given numbers are added. The second of the two problems is not worked out, but the first is worth reproducing. We must take the particular figures used by Diophantus, namely $a=2, b=6$. We have then to divide 9 into two squares such that one of them lies between 2 and 3. Take two squares lying between 2 and 3, say $\frac{25}{16}, \frac{9}{4}$. We have then to find a square ξ^2 lying between them; if we can do this, we can make $9-\xi^2$ a square, and so solve the problem.

Put $9-\xi^2 = (3-m\xi)^2$, say, so that $\xi = 6m/(m^2+1)$; and m has to be determined so that ξ lies between $\frac{1}{2}$ and $\frac{3}{2}$.

$$\text{Therefore} \quad \frac{17}{12} < \frac{6m}{m^2+1} < \frac{19}{12}.$$

Diophantus, as we have seen, finds *a fortiori* integral limits for m by solving these inequalities, making m not greater than $\frac{9}{7}$ and not less than $\frac{9}{8}$ (see pp. 463-5 above). He then takes $m=3\frac{1}{2}$ and puts $9-\xi^2 = (3-3\frac{1}{2}\xi)^2$, which gives $\xi = \frac{8}{5}$.]

$$13. x+y+z=a, y+z=u^2, z+x=v^2, x+y=w^2.$$

$$14. x+y+z+w=a, x+y+z=s^2, y+z+w=t^2,$$

$$z+w+x=u^2, w+x+y=v^2.$$

[The method is the same.]

$$21. x^2y^2z^2+x^2=u^2, x^2y^2z^2+y^2=v^2, x^2y^2z^2+z^2=w^2.$$

$$22. \omega^2y^2z^2-x^2=u^2, x^2y^2z^2-y^2=v^2, \omega^2y^2z^2-z^2=w^2.$$

$$23. x^2-x^2y^2z^2=u^2, y^2-x^2y^2z^2=v^2, z^2-\omega^2y^2z^2=w^2.$$

[Solved by means of right-angled triangles in rational numbers.]

$$\left\{ \begin{array}{l} \text{V. 24. } y^2 z^2 + 1 = u^2, \quad z^2 x^2 + 1 = v^2, \quad x^2 y^2 + 1 = w^2. \\ \text{V. 25. } y^2 z^2 - 1 = u^2, \quad z^2 x^2 - 1 = v^2, \quad x^2 y^2 - 1 = w^2. \\ \text{V. 26. } 1 - y^2 z^2 = u^2, \quad 1 - z^2 x^2 = v^2, \quad 1 - x^2 y^2 = w^2. \end{array} \right.$$

[These reduce to the preceding set of three problems.]

$$\left\{ \begin{array}{l} \text{V. 27. } y^2 + z^2 + a = u^2, \quad z^2 + x^2 + a = v^2, \quad x^2 + y^2 + a = w^2. \\ \text{V. 28. } y^2 + z^2 - a = u^2, \quad z^2 + x^2 - a = v^2, \quad x^2 + y^2 - a = w^2. \end{array} \right.$$

V. 30. $mx + ny = u^2, \quad u^2 + a = (x + y)^2.$

[This problem is enunciated thus. 'A man buys a certain number of measures of wine, some at 8 drachmas, some at 5 drachmas each. He pays for them a *square* number of drachmas; and if 60 is added to this number, the result is a square, the side of which is equal to the whole number of measures. Find the number bought at each price.'

Let ξ = the whole number of measures; therefore $\xi^2 - 60$ was the number of drachmas paid, and $\xi^2 - 60 =$ a square, say $(\xi - m)^2$; hence $\xi = (m^2 + 60)/2m$.

Now $\frac{1}{2}$ of the price of the five-drachma measures $+ \frac{1}{2}$ of that of the eight-drachma measures $= \xi$; therefore $\xi^2 - 60$, the total price, has to be divided into two parts such that $\frac{1}{2}$ of one $+ \frac{1}{2}$ of the other $= \xi$.

We cannot have a real solution of this unless

$$\xi > \frac{1}{2}(\xi^2 - 60) \text{ and } < \frac{1}{2}(\xi^2 - 60);$$

therefore

$$5\xi < \xi^2 - 60 < 8\xi.$$

Diophantus concludes, as we have seen (p. 464 above), that ξ is not less than 11 and not greater than 12.

Therefore, from above, since $\xi = (m^2 + 60)/2m$,

$$22m < m^2 + 60 < 24m;$$

and Diophantus concludes that m is not less than 19 and not greater than 21. He therefore puts $m = 20$.

Therefore $\xi = (m^2 + 60)/2m = 11\frac{1}{2}$, $\xi^2 = 132\frac{1}{4}$, and $\xi^2 - 60 = 72\frac{1}{4}$.

We have now to divide $72\frac{1}{4}$ into two parts such that $\frac{1}{2}$ of one part $+ \frac{1}{2}$ of the other $= 11\frac{1}{2}$.

Let the first part $= 5z$; therefore $\frac{1}{8}$ (second part) $= 11\frac{1}{2} - z$, or second part $= 92 - 8z$.

Therefore $5z + 92 - 8z = 72\frac{1}{4}$, and $z = \frac{79}{12}$;

therefore the number of five-drachma measures is $7\frac{9}{2}$ and the number of eight-drachma measures $5\frac{9}{2}$.]

Lemma 2 to VI. 12. $ax^2 + b = u^2$ (where $a + b = c^2$). } (see p. 467
 Lemma to VI. 15. $ax^2 - b = u^2$ (where $ac^2 - b = c^2$). } above.)

II. 15]. $xy + x + y = u^2$, $x + 1 = \frac{v^2}{w^2}(y + 1)$.

II. 16]. $xy - (x + y) = u^2$, $x - 1 = \frac{v^2}{w^2}(y - 1)$.

V. 32]. $x + 1 = \frac{u^2}{v^2}(x - 1)$.

V. 21]. $x^2 + 1 = u^2$, $y^2 + 1 = v^2$, $z^2 + 1 = w^2$.

(viii) Indeterminate analysis of the third degree.

V. 3. $x^2y = u$, $xy = u^3$.

V. 6. $x^3 + y^2 = u^3$, $z^2 + y^2 = v^2$.

V. 7. $x^3 + y^2 = u^2$, $z^2 + y^2 = v^3$.

V. 8. $x + y^3 = u^3$, $x + y = u$.

V. 9. $x + y^3 = u$, $x + y = u^3$.

V. 10. $x^3 + y^3 = x + y$.

V. 11. $x^3 - y^3 = x - y$.

V. 12. $x^3 + y = y^3 + x$.

} the same problem.

(really reducible
to the second
degree.)

[We may give as examples the solutions of IV. 7, IV. 8, IV. 11.

IV. 7. Since $z^2 + y^2 = a$ cube, suppose $z^2 + y^2 = x^3$. To make $x^3 + y^2$ a square, put $x^3 = a^2 + b^2$, $y^2 = 2ab$, which also satisfies $x^3 - y^2 = z^2$. We have then to make $2ab$ a square. Let $a = \xi$, $b = 2\xi$; therefore $a^2 + b^2 = 5\xi^2$, $2ab = 4\xi^2$, $y = 2\xi$, $z = \xi$, and we have only to make $5\xi^2$ a cube. $\xi = 5$, and $x^3 = 125$, $y^2 = 100$, $z^2 = 25$.

IV. 8. Suppose $x = \xi$, $y^3 = m^3 \xi^3$; therefore $u = (m+1)\xi$ must be the side of the cube $m^3 \xi^3 + \xi$, and

$$m^3 \xi^2 + 1 = (m^3 + 3m^2 + 3m + 1) \xi^2.$$

To solve this, we must have $3m^2 + 3m + 1$ (the difference between consecutive cubes) a square. Put

$$3m^2 + 3m + 1 = (1 - nm)^2, \text{ and } m = (3 + 2n)/(n^2 - 3).$$

IV. 11. Assume $x = (m+1)\xi$, $y = m\xi$, and we have to make $(3m^3 + 3m^2 + 1)\xi^2$ equal to 1, i.e. we have only to make $3m^2 + 3m + 1$ a square.]

IV. 18. $x^3 + y = u^3$, $y^2 + x = v^2$.

IV. 24. $x + y = a$, $xy = u^3 - u$.

[$y = a - x$; therefore $ax - x^2$ has to be made a cube minus its side, say $(mx - 1)^3 - (mx - 1)$.

$$\text{Therefore } ax - x^2 = m^3 x^3 - 3m^2 x^2 + 2mx.$$

To reduce this to a simple equation, we have only to put $m = \frac{1}{2}a$.]

IV. 26. $x + y + z = a$, $xyz = \{(x - y) + (x - z) + (y - z)\}^3$.

$$(x > y > z)$$

[The cube $= 8(x - z)^3$. Let $x = (m+1)\xi$, $z = m\xi$, so that $y = 8\xi/(m^2 + m)$, and we have only to contrive that $8/(m^2 + m)$ lies between m and $m+1$. Dioph. takes the first limit $8 > m^3 + m^2$, and puts

$$8 = (m + \frac{1}{3})^3 \text{ or } m^3 + m^2 + \frac{1}{3}m + \frac{1}{27},$$

whence $m = \frac{5}{3}$; therefore $x = \frac{8}{3}\xi$, $y = \frac{8}{5}\xi$, $z = \frac{5}{3}\xi$. Or, multiplying by 15, we have $x = 40\xi$, $y = 27\xi$, $z = 25\xi$. The first equation then gives ξ .]

{ IV. 26. $xy + x = u^3$, $xy + y = v^3$.

{ IV. 27. $xy - x = u^3$, $xy - y = v^3$.

IV. 28. $xy + (x + y) = u^3$, $xy - (x + y) = v^3$.

$$[x + y = \frac{1}{2}(u^3 - v^3), \quad xy = \frac{1}{2}(u^3 + v^3); \text{ therefore}$$

$$(x - y)^2 = \frac{1}{4}(u^3 - v^3)^2 - 2(u^3 + v^3),$$

which latter expression has to be made a square.

Diophantus assumes $u = \xi + 1$, $v = \xi - 1$, whence

$$\frac{1}{4}(6\xi^2 + 2)^2 - 2(2\xi^3 + 6\xi)$$

must be a square, or

$$9\xi^4 - 4\xi^3 + 6\xi^2 - 12\xi + 1 = \text{a square} = (3\xi^2 - 6\xi + 1)^2, \text{ say ;}$$

therefore $32\xi^3 = 36\xi^2$, and $\xi = \frac{9}{8}$. Thus u, v are found, and then x, y .

The second (alternative) solution uses the formula that $\xi(\xi^2 - \xi) + (\xi^2 - \xi) + \xi = \text{a cube}$. Put $x = \xi$, $y = \xi^2 - \xi$, and one condition is satisfied. We then only have to make $\xi(\xi^2 - \xi) - \xi - (\xi^2 - \xi)$ or $\xi^3 - 2\xi^2$ a cube (less than ξ^3), i.e. $\xi^3 - 2\xi^2 = (\frac{1}{2}\xi)^3$, say.]

$$V. 38. (x+y+z)x = \frac{1}{2}u(u+1), (x+y+z)y = v^2,$$

$$(x+y+z)z = w^3, [x+y+z = l^2].$$

[Suppose $x+y+z = \xi^2$; then

$$x = \frac{u(u+1)}{2\xi^2}, y = \frac{v^2}{\xi^2}, z = \frac{w^3}{\xi^2};$$

$$\text{therefore } \xi^4 = \frac{1}{2}u(u+1) + v^2 + w^3.$$

Diophantus puts 8 for w^3 , but we may take any cube, as m^3 ; and he assumes $v^2 = (\xi^2 - 1)^2$, for which we might substitute $(\xi^2 - n^2)^2$. We then have the triangular number $\frac{1}{2}u(u+1) = 2n^2\xi^2 - n^4 - m^3$. Since 8 times a triangular number plus 1 gives a square,

$$16n^2\xi^2 - 8n^4 - 8m^3 + 1 = \text{a square} = (4n\xi - k)^2, \text{ say,}$$

and the problem is solved.]

$$V. 15. (x+y+z)^3 + x = u^3, (x+y+z)^3 + y = v^4,$$

$$(x+y+z)^3 + z = w^3.$$

$$[\text{Let } x+y+z = \xi, u^3 = m^3\xi^3, v^4 = n^3\xi^3, w^3 = p^3\xi^3;$$

$$\text{therefore } \xi = \{(m^3-1) + (n^3-1) + (p^3-1)\}\xi^3;$$

and we have to find three cubes m^3, n^3, p^3 such that $m^3 + n^3 + p^3 - 3 = \text{a square}$. Diophantus assumes as the sides of the cubes $(k+1), (2-k), 2$; this gives

$9k^2 - 9k + 14 = \text{a square} = (3k - l)^2$, say; and k is found.

Retracing our steps, we find ξ and therefore x, y, z .]

$$\text{V. 16. } (x+y+z)^3 - x = u^3, (x+y+z)^3 - y = v^3,$$

$$(x+y+z)^3 - z = w^3.$$

$$\text{V. 17. } x - (x+y+z)^3 = u^3, y - (x+y+z)^3 = v^3,$$

$$z - (x+y+z)^3 = w^3.$$

$$\text{V. 18. } x+y+z = t^2, (x+y+z)^3 + x = u^2, (x+y+z)^3 + y = v^2,$$

$$(x+y+z)^3 + z = w^2.$$

[Put $x+y+z = \xi^2$, $x = (p^2-1)\xi^0$, $y = (q^2-1)\xi^0$, $z = (r^2-1)\xi^0$, whence $\xi^2 = (p^2-1+q^2-1+r^2-1)\xi^0$, so that $p^2-1+q^2-1+r^2-1$ must be made a fourth power. Diophantus assumes $p^2 = (m^2-1)^2$, $q^2 = (m+1)^2$, $r^2 = (m-1)^2$, since $m^4 - 2m^2 + m^2 + 2m + m^2 - 2m = m^4$.]

$$\text{V. 19. } x+y+z = t^2, (x+y+z)^3 - x = u^2,$$

$$(x+y+z)^3 - y = v^2, (x+y+z)^3 - z = w^2.$$

$$\text{V. 19a. } x+y+z = t^2, x - (x+y+z)^3 = u^2,$$

$$y - (x+y+z)^3 = v^2, z - (x+y+z)^3 = w^2.$$

$$\text{V. 19. b, c. } x+y+z = a, (x+y+z)^3 \pm x = u^2,$$

$$(x+y+z)^3 \pm y = v^2, (x+y+z)^3 \pm z = w^2.$$

$$\text{V. 20. } x+y+z = \frac{1}{m}, x - (x+y+z)^3 = u^2,$$

$$y - (x+y+z)^3 = v^2, z - (x+y+z)^3 = w^2.$$

$$[\text{IV. 8.}] \quad x - y = 1, x^3 - y^3 = u^2.$$

$$[\text{IV. 9, 10.}] \quad x^3 + y^3 = \frac{u^2}{v^2} (x+y).$$

$$[\text{IV. 11.}] \quad x^3 - y^3 = \frac{u^2}{v^2} (x-y).$$

$$[\text{V. 15.}] \quad x^3 + y^3 + z^3 - 3 = u^2.$$

$$[\text{V. 16.}] \quad 3 - (x^3 + y^3 + z^3) = u^2.$$

$$[\text{V. 17.}] \quad x^3 + y^3 + z^3 + 3 = u^2.$$

(ix) Indeterminate analysis of the fourth degree.

V. 29. $w^4 + y^4 + z^4 = u^2$.

['Why', says Fermat, 'did not Diophantus seek two fourth powers such that their sum is a square. This problem is, in fact, impossible, as by my method I am able to prove with all rigour.' No doubt Diophantus knew this truth empirically. Let $w^2 = \xi^2$, $y^2 = p^2$, $z^2 = q^2$. Therefore $\xi^4 + p^4 + q^4 =$ a square $= (\xi^2 - r)^2$, say; therefore $\xi^2 = (r^2 - p^4 - q^4)/2r$, and we have to make this expression a square.

Diophantus puts $r = p^2 + 4$, $q^2 = 4$, so that the expression reduces to $8p^2/(2p^2 + 8)$ or $4p^2/(p^2 + 4)$. To make this a square, let $p^2 + 4 = (p + 1)^2$, say; therefore $p = 1\frac{1}{2}$, and $p^2 = 2\frac{1}{4}$, $q^2 = 4$, $r = 6\frac{1}{4}$; or (multiplying by 4) $p^2 = 9$, $q^2 = 16$, $r = 25$, which solves the problem.]

V. 18]. $w^2 + y^2 + z^2 - 3 = u^4$.

(See above under V. 18.)

(x) Problems of constructing right-angled triangles with sides in rational numbers and satisfying various other conditions.

[I shall in all cases call the hypotenuse z , and the other two sides w, y , so that the condition $w^2 + y^2 = z^2$ applies in all cases, in addition to the other conditions specified.]

Lemma to V. 7]. $wy = w_1y_1 = w_2y_2$.

VI. 1. $z - x = u^3$, $z - y = v^3$.

[Form a right-angled triangle from ξ, m , so that $z = \xi^2 + m^2$, $w = 2m\xi$, $y = \xi^2 - m^2$; thus $z - y = 2m^2$, and, as this must be a cube, we put $m = 2$; therefore $z - x = \xi^2 - 4\xi + 4$ must be a cube, or $\xi - 2 =$ a cube, say n^3 , and $\xi = n^3 + 2$.]

VI. 2. $z + x = u^3$, $z + y = v^3$.

VI. 3. $\frac{1}{2}xy + a = u^2$.

[Suppose the required triangle to be $h\xi$, $p\xi$, $b\xi$; therefore $\frac{1}{2}pb\xi^2 + a = \text{a square} = n^2\xi^2$, say, and the ratio of a to $n^2 - \frac{1}{2}pb$ must be the ratio of a square to a square. To find n , p , b so as to satisfy this condition, form a right-angled triangle from m , $\frac{1}{m}$,

i.e. $\left(m^2 + \frac{1}{m^2}, 2, m^2 - \frac{1}{m^2}\right)$;

therefore $\frac{1}{2}pb = m^2 - \frac{1}{m^2}$. Assume $n^2 = \left(m + \frac{2a}{m}\right)^2$;

therefore $n^2 - \frac{1}{2}pb = 4a + \frac{4a^2 + 1}{m^2}$; and $\left(4a + \frac{4a^2 + 1}{m^2}\right)/a$,

or $4a^2 + \frac{a(4a^2 + 1)}{m^2}$, has to be made a square. Put

$4a^2m^2 + a(4a^2 + 1) = (2am + k)^2$, and we have a solution. Diophantus has $a = 5$, leading to $100m^2 + 505 = \text{a square} = (10m + 5)^2$, say, which gives $m = \frac{24}{5}$ and $n = \frac{41}{5}$. h , p , b are thus determined in such a way that $\frac{1}{2}pb\xi^2 + a = n^2\xi^2$ gives a rational solution.]

VI. 4. $\frac{1}{2}xy - a = u^2$.

VI. 5. $a - \frac{1}{2}xy = u^2$.

VI. 6. $\frac{1}{2}xy + x = a$.

[Assume the triangle to be $h\xi$, $p\xi$, $b\xi$, so that $\frac{1}{2}pb\xi^2 + p\xi = a$, and for a rational solution of this equation we must have $(\frac{1}{2}p)^2 + a(\frac{1}{2}pb)$ a square. Diophantus assumes $p = 1$, $b = m$, whence $\frac{1}{2}am + \frac{1}{4}$ or $2am + 1 = \text{a square}$.

But, since the triangle is rational, $m^2 + 1 = \text{a square}$.

That is, we have a double equation. Difference $= m^2 - 2am = m(m - 2a)$. Put

$$2am + 1 = \left\{\frac{1}{2}(m - m - 2a)\right\}^2 = a^2, \text{ and } m = (a^2 - 1)/2a.$$

The sides of the auxiliary triangle are thus determined in such a way that the original equation in ξ is solved rationally.]

VI. 7. $\frac{1}{2}xy - x = a$.

$$8. \frac{1}{2}xy + (x + y) = a.$$

$$9. \frac{1}{2}xy - (x + y) = a.$$

[With the same assumptions we have in these cases to make $\{\frac{1}{2}(p+b)\}^2 + a(\frac{1}{2}pb)$ a square. Diophantus assumes as before 1, m for the values of p , b , and obtains the double equation

$$\left. \begin{aligned} \frac{1}{4}(m+1)^2 + \frac{1}{2}am &= \text{square} \\ m^2 + 1 &= \text{square} \end{aligned} \right\},$$

$$\text{or } \left. \begin{aligned} m^2 + (2a+2)m + 1 &= \text{square} \\ m^2 + 1 &= \text{square} \end{aligned} \right\},$$

solving in the usual way.]

$$10. \frac{1}{2}xy + x + z = a.$$

$$11. \frac{1}{2}xy - (x + z) = a.$$

[In these cases the auxiliary right-angled triangle has to be found such that

$$\{\frac{1}{2}(h+p)\}^2 + a(\frac{1}{2}pb) = \text{a square}.$$

Diophantus assumes it formed from 1, $m+1$; thus

$$\frac{1}{4}(h+p)^2 = \frac{1}{4}\{m^2 + 2m + 2 + m^2 + 2m\}^2 = (m^2 + 2m + 1)^2,$$

$$\text{and } a(\frac{1}{2}pb) = a(m+1)(m^2 + 2m).$$

Therefore

$$m^4 + (a+4)m^3 + (3a+6)m^2 + (2a+4)m + 1$$

$$= \text{a square}$$

$$= \{1 + (a+2)m - m^2\}^2, \text{ say;}$$

and m is found.]

$$\text{na 1 to VI. 12. } x = u^2, x - y = v^2, \frac{1}{2}xy + y = w^2.$$

$$2. \frac{1}{2}xy + x = u^2, \frac{1}{2}xy + y = v^2.$$

$$3. \frac{1}{2}xy - x = u^2, \frac{1}{2}xy - y = v^2.$$

[These problems and the two following are interesting, but their solutions run to some length; therefore only one case can here be given. We will take VI. 12 with its Lemma 1.

Lemma 1. If a rational right-angled triangle be formed from m, n , the perpendicular sides are $2mn, m^2 - n^2$. We will suppose the greater of the two to be $2mn$. The first two relations are satisfied by making $m = 2n$. Form, therefore, a triangle from $\xi, 2\xi$. The third condition then gives $6\xi^4 + 3\xi^2 = \text{a square}$ or $6\xi^2 + 3 = \text{a square}$. One solution is $\xi = 1$ (and there are an infinite number of others to be found by means of it). If $\xi = 1$, the triangle is formed from 1, 2.

VI. 12. Suppose the triangle to be $(h\xi, b\xi, p\xi)$. Then $(\frac{1}{2}pb)\xi^2 + p\xi = \text{a square} = (k\xi)^2$, say, and $\xi = p / (k^2 - \frac{1}{2}pb)$. This value must be such as to make $(\frac{1}{2}pb)\xi^2 + b\xi$ a square also. By substitution of the value of ξ we get

$$\{bpk^2 + \frac{1}{2}p^2b(p-b)\} / (k^2 - \frac{1}{2}pb)^2;$$

so that $bpk^2 + \frac{1}{2}p^2b(p-b)$ must be a square; or, if p , the greater perpendicular, is made a square number, $bk^2 + \frac{1}{2}pb(p-b)$ has to be made a square. This by Lemma 2 (see p. 467 above) can be made a square if $b + \frac{1}{2}pb(p-b)$ is a square. *How to solve these problems*, says Diophantus, *is shown in the Lemmas*. It is not clear how they were applied, but, in fact, his solution is such as to make $p, p-b$, and $b + \frac{1}{2}pb$ all squares, namely $b = 3, p = 4, h = 5$.

Accordingly, putting for the original triangle $3\xi, 4\xi, 5\xi$, we have

$$\left. \begin{aligned} 6\xi^2 + 4\xi &= \text{a square} \\ 6\xi^2 + 3\xi &= \text{a square} \end{aligned} \right\}.$$

Assuming $6\xi^2 + 4\xi = m^2\xi^2$, we have $\xi = 4 / (m^2 - 6)$, and the second condition gives

$$\frac{96}{m^4 - 12m^2 + 36} + \frac{12}{m^2 - 6} = \text{a square},$$

$$\text{or} \quad 12m^2 + 24 = \text{a square}.$$

This can be solved, since $m = 1$ satisfies it (Lemma 2).

A solution is $m^2 = 25$, whence $\xi = \frac{4}{16}$.]

$$\left\{ \begin{aligned} \text{VI. 14. } \frac{1}{2}xy - z &= u^2, \quad \frac{1}{2}xy - x = v^2. \end{aligned} \right.$$

$$\left\{ \begin{aligned} \text{VI. 15. } \frac{1}{2}xy + z &= u^2, \quad \frac{1}{2}xy + x = v^2. \end{aligned} \right.$$

[The auxiliary right-angled triangle in this case must be such that

$$m^2 hp - \frac{1}{2} pb \cdot p(h-p) \text{ is a square.}$$

If, says Diophantus (VI. 14), we form a triangle from the numbers X_1, X_2 and suppose that $p = 2X_1X_2$, and if we then divide out by $(X_1 - X_2)^2$, which is equal to $h - p$, we must find a square $k^2 [= m^2 / (X_1 - X_2)^2]$ such that $k^2 hp - \frac{1}{2} pb \cdot p$ is a square.

The problem, says Diophantus, can be solved if X_1, X_2 are 'similar plane numbers' (numbers such as $ab, \frac{m^2}{n^2} ab$).

This is stated without proof, but it can easily be verified that, if $k^2 = X_1X_2$, the expression is a square. Dioph. takes 4, 1 as the numbers, so that $k^2 = 4$. The equation for m becomes

$$8 \cdot 17m^2 - 4 \cdot 15 \cdot 8 \cdot 9 = \text{a square,}$$

$$\text{or } 136m^2 - 4320 = \text{a square.}$$

The solution $m^2 = 36$ (derived from the fact that

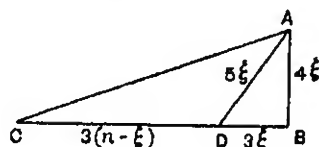
$$k^2 = m^2 / (X_1 - X_2)^2, \text{ or } 4 = m^2 / 3^2)$$

satisfies the condition that

$$m^2 hp - \frac{1}{2} pb \cdot p(h-p) \text{ is a square.}]$$

$$16. \xi + \eta = x, \xi / \eta = y / z.$$

[To find a rational right-angled triangle such that the number representing the (portion intercepted within the triangle of the) bisector of an acute angle is rational.



Let the bisector be 5ξ , the segment BD of the base 3ξ , so that the perpendicular is 4ξ .

Let $CB = 3n$. Then $AC : AB = CD : DB$,

so that $AC = 4(n - \xi)$. Therefore (Eucl. I. 47)

$$16(n^2 - 2n\xi + \xi^2) = 16\xi^2 + 9n^2,$$

so that $\xi = 7n^2/32n = \frac{7}{32}n$. [Dioph. has $n = 1$.]

VI. 17. $\frac{1}{2}xy + z = u^3$, $x + y + z = v^3$.

[Let ξ be the area $\frac{1}{2}xy$, and let $z = k^2 - \xi$. Since $xy = 2\xi$, suppose $x = 2$, $y = \xi$. Therefore $2 + k^2$ must be a cube. As we have seen (p. 475), Diophantus takes $(m-1)^3$ for the cube and $(m+1)^3$ for k^2 , giving $m^3 - 3m^2 + 3m - 1 = m^2 + 2m + 3$, whence $m = 4$. Therefore $k = 5$, and we assume $\frac{1}{2}xy = \xi$, $z = 25 - \xi$, with $x = 2$, $y = \xi$ as before. Then we have to make $(25 - \xi)^2 = 4 + \xi^2$, and $\xi = \frac{241}{80}$.]

VI. 18. $\frac{1}{2}xy + z = u^3$, $x + y + z = v^3$.

VI. 19. $\frac{1}{2}xy + x = u^2$, $x + y + z = v^3$.

[Here a right-angled triangle is formed from one odd number, say $2\xi + 1$, according to the Pythagorean formula $m^2 + \{\frac{1}{2}(m^2 - 1)\}^2 = \{\frac{1}{2}(m^2 + 1)\}^2$, where m is an odd number. The sides are therefore $2\xi + 1$, $2\xi^2 + 2\xi$, $2\xi^2 + 2\xi + 1$. Since the perimeter = a cube,

$$4\xi^2 + 6\xi + 2 = (4\xi + 2)(\xi + 1) = \text{a cube.}$$

Or, if we divide the sides by $\xi + 1$, $4\xi + 2$ has to be made a cube.

$$\text{Again } \frac{1}{2}xy + x = \frac{2\xi^3 + 3\xi^2 + \xi}{(\xi + 1)^2} + \frac{2\xi + 1}{\xi + 1} = \text{a square,}$$

which reduces to $2\xi + 1 = \text{a square.}$

But $4\xi + 2$ is a cube. We therefore put 8 for the cube, and $\xi = 1\frac{1}{2}$.]

VI. 20. $\frac{1}{2}xy + x = u^3$, $x + y + z = v^2$.

VI. 21. $x + y + z = u^2$, $\frac{1}{2}xy + (x + y + z) = v^3$.

[Form a right-angled triangle from ξ , 1, i.e. $(2\xi, \xi^2 - 1, \xi^2 + 1)$. Then $2\xi^2 + 2\xi$ must be a square, and $\xi^3 + 2\xi^2 + \xi$

cube. Put $2\xi^2 + 2\xi = m^2\xi^2$, so that $\xi = 2/(m^2 - 2)$, and we have to make

$$\frac{8}{(m^2 - 2)^3} + \frac{8}{(m^2 - 2)^2} + \frac{2}{m^2 - 2}, \text{ or } \frac{2m^4}{(m^2 - 2)^3}, \text{ a cube.}$$

Take $2m$ a cube $= n^3$, so that $2m^4 = m^3n^3$, and $n = \frac{1}{2}n^3$; therefore $\xi = \frac{8}{n^3 - 8}$, and ξ must be made greater than 1, in order that $\xi^2 - 1$ may be positive.

Therefore $8 < n^3 < 16$;

this is satisfied by $n^3 = \frac{7 \cdot 2^9}{6^4}$ or $n^3 = \frac{2^7}{8}$, and $m = \frac{7^{\frac{1}{3}}}{18}$.]

$$2. \ x + y + z = w^3, \ \frac{1}{2}xy + (x + y + z) = v^2.$$

[(1) First seek a rational right-angled triangle such that its perimeter and its area are given numbers, any p, m .

Let the perpendiculars be $\frac{1}{\xi}, 2m\xi$; therefore the hypo-

tenuse $= p - \frac{1}{\xi} - 2m\xi$, and (Eucl. I. 47)

$$\frac{1}{\xi^2} + 4m^2\xi^2 + (p^2 + 4m) - \frac{2p}{\xi} - 4mp\xi = \frac{1}{\xi^2} + 4m^2\xi^2,$$

$$\text{or } p^2 + 4m = 4mp\xi + \frac{2p}{\xi},$$

that is, $(p^2 + 4m)\xi = 4mp\xi^2 + 2p$.

(2) In order that this may have a rational solution,

$$\left\{ \frac{1}{2}(p^2 + 4m) \right\}^2 - 8p^2m \text{ must be a square,}$$

$$\text{e. } 4m^2 - 6p^2m + \frac{1}{4}p^4 = \text{a square,}$$

$$\text{or } m^2 - \frac{3}{2}p^2m + \frac{1}{16}p^4 = \text{a square} \left. \vphantom{\begin{matrix} m^2 - \frac{3}{2}p^2m + \frac{1}{16}p^4 \\ \text{Also, by the second condition, } m + p \end{matrix}} \right\} = \text{a square}$$

Also, by the second condition, $m + p = \text{a square}$

To solve this, we must take for p some number which is both a square and a cube (in order that it may be possible, by multiplying the second equation by some square, to make the constant term equal to the constant

term in the first). Diophantus takes $p = 64$, making the equations

$$\left. \begin{aligned} m^2 - 6144m + 1048576 &= \text{a square} \\ m + 64 &= \text{a square} \end{aligned} \right\}.$$

Multiplying the second by 16384, and subtracting the two expressions, we have as the difference $m^2 - 22528m$.

Diophantus observes that, if we take m , $m - 22528$ as the factors, we obtain $m = 7680$, an impossible value for the area of a right-angled triangle of perimeter $p = 64$.

We therefore take as factors $11m$, $\frac{1}{11}m - 2048$, and, equating the square of half the difference ($= \frac{99}{11}m + 1024$) to $16384m + 1048576$, we have $m = 3\frac{9}{2}\frac{2}{3}^4$.

(3) Returning to the original problem, we have to substitute this value for m in

$$(64 - \frac{1}{\xi} - 2m\xi)^2 = \frac{1}{\xi^2} + 4m^2\xi^2,$$

and we obtain

$$78848\xi^2 - 8432\xi + 225 = 0,$$

the solution of which is rational, namely $\xi = \frac{25}{448}$ (or $\frac{9}{176}$). Diophantus naturally takes the first value, though the second gives the same triangle.]

VI. 23. $z^2 = u^2 + u$, $z^2/x = v^3 + v$.

VI. 24. $z = u^3 + u$, $x = v^3 - v$, $y = w^3$.

[VI. 6, 7]. $(\frac{1}{2}x)^2 + \frac{1}{2}mxy = u^2$.

[VI. 8, 9]. $\{\frac{1}{2}(x+y)\}^2 + \frac{1}{2}mxy = u^2$.

[VI. 10, 11]. $\{\frac{1}{2}(z+x)\}^2 + \frac{1}{2}mxy = u^2$.

[VI. 12.] $y + (x-y) \cdot \frac{1}{2}xy = u^2$, $x = v^2$. ($x > y$.)

[VI. 14, 15]. $u^2zx - \frac{1}{2}xy \cdot x(z-x) = v^2$. ($u^2 < \text{or} > \frac{1}{2}xy$.)

The treatise on Polygonal Numbers.

The subject of Polygonal Numbers on which Diophantus also wrote is, as we have seen, an old one, going back to the

Pythagoreans, while Philippus of Opus and Spousippus carried the tradition. Hypsicles (about 170 B.C.) is twice mentioned by Diophantus as the author of a 'definition' of polygonal number which, although it does not in terms mention any polygonal number beyond the pentagonal, amounts to saying that the n th a -gon (1 counting as the first) is

$$\frac{1}{2}n\{2 + (n-1)(a-2)\}.$$

At Smyrna, Nicomachus and Iamblichus all devote much space to polygonal numbers. Nicomachus in particular gives various rules for transforming triangles into squares, squares into pentagons, &c.

If we put two consecutive triangles together, we get a square. The fact

$$\frac{1}{2}(n-1)n + \frac{1}{2}n(n+1) = n^2.$$

A regular pentagon is obtained from a square by adding to it a triangle the side of which is 1 less than that of the square; a regular hexagon from a pentagon by adding a triangle the side of which is 1 less than that of the pentagon, and so on. The fact

$$\begin{aligned} & \frac{1}{2}\{2 + (n-1)(a-2)\} + \frac{1}{2}(n-1)n \\ &= \frac{1}{2}n[2 + (n-1)\{(a+1)-2\}]. \end{aligned}$$

Nicomachus sets out the first triangles, squares, pentagons, hexagons and heptagons in a diagram thus:

Triangles	1	3	6	10	15	21	28	36	45	55,
Squares	1	4	9	16	25	36	49	64	81	100,
Pentagons	1	5	12	22	35	51	70	92	117	145,
Hexagons	1	6	15	28	45	66	91	120	153	190,
Heptagons	1	7	18	34	65	81	112	148	189	235,

Nicomachus observes that:

Each polygon is equal to the polygon immediately above it in the diagram *plus* the triangle with 1 less in its side, i.e. the triangle in the preceding column.

4. The vertical columns are in arithmetical progression, the common difference being the triangle in the preceding column.

Plutarch, a contemporary of Nicomachus, mentions another method of transforming triangles into squares. *Every triangular number taken eight times and then increased by 1 gives a square.*

$$\text{In fact,} \quad 8 \cdot \frac{1}{2} n(n+1) + 1 = (2n+1)^2.$$

Only a fragment of Diophantus's treatise *On Polygonal Numbers* survives. Its character is entirely different from that of the *Arithmetica*. The method of proof is strictly geometrical, and has the disadvantage, therefore, of being long and involved. He begins with some preliminary propositions of which two may be mentioned. Prop. 3 proves that, if a be the first and l the last term in an arithmetical progression of n terms, and if s is the sum of the terms, $2s = n(l+a)$. Prop. 4 proves that, if $1, 1+b, 1+2b, \dots, 1+(n-1)b$ be an A. P., and s the sum of the terms,

$$2s = n \{2 + (n-1)b\}.$$

The main result obtained in the fragment as we have it is a generalization of the formula $8 \cdot \frac{1}{2} n(n+1) + 1 = (2n+1)^2$. Prop. 5 proves the fact stated in Hypsicles's definition and also (the generalization referred to) that

$$8P(a-2) + (a-4)^2 = \text{a square,}$$

where P is any polygonal number with a angles.

It is also proved that, if P be the n th a -gonal number (1 being the first),

$$8P(a-2) + (a-4)^2 = \{2 + (2n-1)(a-2)\}^2.$$

Diophantus deduces rules as follows.

1. To find the number from its side.

$$P = \frac{\{2 + (2n-1)(a-2)\}^2 - (a-4)^2}{8(a-2)}.$$

2. To find the side from the number.

$$n = \frac{1}{2} \left(\frac{\sqrt{\{8P(a-2) + (a-4)^2\}} - 2}{a-2} + 1 \right).$$

The last proposition, which breaks off in the middle, is :

Given a number, to find in how many ways it can be polygonal.

The proposition begins in a way which suggests that Diophantus first proved geometrically that, if

$$8P(a-2) + (a-4)^2 = \{2 + (2n-1)(a-2)\}^2,$$

$$2P = n\{2 + (n-1)(a-2)\}.$$

Wertheim (in his edition of Diophantus) has suggested a variation of the complete proof of this proposition, and I have shown (in my edition) how the proof can be made shorter. Wertheim adds an investigation of the main problem, but no doubt opinions will continue to differ as to whether Diophantus actually solved it.

COMMENTATORS AND BYZANTINES

WE have come to the last stage of Greek mathematics; it only remains to include in a last chapter references to commentators of more or less note who contributed nothing original but have preserved, among observations and explanations obvious or trivial from a mathematical point of view, valuable extracts from works which have perished, or historical allusions which, in the absence of original documents, are precious in proportion to their rarity. Nor must it be forgotten that in several cases we probably owe to the commentators the fact that the masterpieces of the great mathematicians have survived, wholly or partly, in the original Greek or at all. This may have been the case even with the works of Archimedes on which Eutocius wrote commentaries. It was no doubt these commentaries which aroused in the school of Isidorus of Miletus (the colleague of Anthemius as architect of Saint Sophia at Constantinople) a new interest in the works of Archimedes and caused them to be sought out in the various libraries or wherever they had lain hid. This revived interest apparently had the effect of evoking new versions of the famous works commented upon in a form more convenient for the student, with the Doric dialect of the original eliminated; this translation of the Doric into the more familiar dialect was systematically carried out in those books only which Eutocius commented on, and it is these versions which alone survive. Again, Eutocius's commentary on Apollonius's *Conics* is extant for the first four Books, and it is probably owing to their having been commented on by Eutocius, as well as to their being more elementary than the rest, that these four Books alone

live in Greek. Tannery, as we have seen, conjectured, in like manner, the first six of the thirteen Books of Pappus's *Arithmetica* survive because Hypatia wrote commentaries on these Books only and did not reach the others.

The first writer who calls for notice in this chapter is one who was rather more than a commentator in so far as he wrote a couple of treatises to supplement the *Conics* of Apollonius, I mean SERENUS. Serenus came from Antinocia, Antinoupolis, a city in Egypt founded by Hadrian (A.D. 118-138). His date is uncertain, but he most probably belonged to the fourth century A.D., and came between Pappus and Theon of Alexandria. He tells us himself that he wrote a commentary on the *Conics* of Apollonius.¹ This has been lost, and, apart from a certain proposition 'of Serenus the philosopher, from the Lemmas' preserved in certain manuscripts of Theon of Smyrna (to the effect that, if a number of parallel lineal angles be subtended at a point on a diameter of a circle which is not the centre, by equal arcs of that circle, the angle nearer to the centre is always less than the angle more remote), we have only the two small treatises by him entitled *the Section of a Cylinder* and *On the Section of a Cone*. These works came to be connected, from the seventh century onwards, with the *Conics* of Apollonius, on account of the similarity of the subjects, and this no doubt accounts for their survival. They were translated into Latin by Commandinus in 1566; the first Greek text was brought out by Halley along with his Apollonius (Oxford 1710), and we now have the definitive text edited by Heiberg (Teubner 1896).

(a) *On the Section of a Cylinder.*

On the occasion and the object of the tract *On the Section of a Cylinder* are stated in the preface. Serenus observes that many persons who were students of geometry were under the erroneous impression that the oblique section of a cylinder was different from the oblique section of a cone known as an ellipse, whereas it is of course the same curve. Hence he makes it necessary to establish, by a regular geometrical

¹ Serenus, *Opuscula*, ed. Heiberg, p. 52, 25-6.

proof, that the said oblique sections cutting all the generators are equally ellipses whether they are sections of a cylinder or of a cone. He begins with 'a more general definition' of a cylinder to include any oblique circular cylinder. 'If in two equal and parallel circles which remain fixed the diameters, while remaining parallel to one another throughout, are moved round in the planes of the circles about the centres, which remain fixed, and if they carry round with them the straight line joining their extremities on the same side until they bring it back again to the same place, let the surface described by the straight line so carried round be called a *cylindrical surface*.' The *cylinder* is the figure contained by the parallel circles and the cylindrical surface intercepted by them; the parallel circles are the *bases*, the *axis* is the straight line drawn through their centres; the generating straight line in any position is a *side*. Thirty-three propositions follow. Of these Prop. 6 proves the existence in an oblique cylinder of the parallel circular sections subcontrary to the series of which the bases are two, Prop. 9 that the section by any plane not parallel to that of the bases or of one of the subcontrary sections but cutting all the generators is not a circle; the next propositions lead up to the main results, namely those in Props. 14 and 16, where the said section is proved to have the property of the ellipse which we write in the form

$$QV^2 : PV \cdot P'V = CD^2 : CP^2,$$

and in Prop. 17, where the property is put in the Apollonian form involving the *latus rectum*, $QV^2 = PV \cdot VR$ (see figure on p. 137 above), which is expressed by saying that the square on the semi-ordinate is equal to the rectangle applied to the *latus rectum* PL , having the abscissa PV as breadth and falling short by a rectangle similar to the rectangle contained by the diameter PP' and the *latus rectum* PL (which is determined by the condition $PL \cdot PP' = DD'^2$ and is drawn at right angles to PV). Prop. 18 proves the corresponding property with reference to the conjugate diameter DD' and the corresponding *latus rectum*, and Prop. 19 gives the main property in the form $QV^2 : PV \cdot P'V = Q'V'^2 : P'V' \cdot P''V'$. Then comes the proposition that 'it is possible to exhibit a cone and a cylinder which are alike cut in one and the same ellipse' (Prop. 20).

Serenus then solves such problems as these: Given a cone (or cylinder) and an ellipse on it, to find the cylinder (cone) which is cut in the same ellipse as the cone (cylinder) (Props. 21, 22); given a cone (cylinder), to find a cylinder (cone) and to cut both by one and the same plane so that the sections thus made shall be similar ellipses (Props. 23, 24). Props. 27, 28 deal with similar elliptic sections of a scalene cylinder and cone; there are two pairs of infinite sets of these similar to any one given section, the first pair being those which are parallel and subcontrary respectively to the given section, the other pair subcontrary to one another but not to either of the other sets and having the conjugate diameter occupying the corresponding place to the transverse in the other sets, and vice versa.

In the propositions (29-33) from this point to the end of the book Serenus deals with what is really an optical problem. It is introduced by a remark about a certain geometer, Peithon by name, who wrote a tract on the subject of parallels. Peithon, not being satisfied with Euclid's treatment of parallels, thought to define parallels by means of an illustration, observing that parallels are such lines as are shown on a wall or a roof by the shadow of a pillar with a light behind it. This definition, it appears, was generally ridiculed; and Serenus seeks to rehabilitate Peithon, who was his friend, by showing that his statement is after all mathematically sound. He therefore proves, with regard to the cylinder, that, if any number of rays from a point outside the cylinder are drawn touching it on both sides, all the rays pass through the sides of a parallelogram (a section of the cylinder parallel to the axis)—Prop. 29—and if they are produced farther to meet any other plane parallel to that of the parallelogram the points in which they meet the plane will lie on two parallel lines (Prop. 30); he adds that the lines will not *seem* parallel (*vide* Euclid's *Optics*, Prop. 6). The problem about the rays touching the surface of a cylinder suggests the similar one about any number of rays from an external point touching the surface of a cone; these meet the surface in points on a triangular section of the cone (Prop. 32) and, if produced to meet a plane parallel to that of the triangle, meet that plane in points forming a similar triangle

(Prop. 33). Prop. 31 preceding these propositions is a particular case of the constancy of the anharmonic ratio of a pencil of four rays. If two sides AB, AC of a triangle meet a transversal through D , an external point, in E, F and another ray AG between AB and AC cuts DEF in a point G such that $ED:DF = EG:GF$, then any other transversal through D meeting AB, AG, AC in K, L, M is also divided harmonically, i.e. $KD:DM = KL:LM$. To prove the succeeding propositions, 32 and 33, Serenus uses this proposition and a reciprocal of it combined with the harmonic property of the pole and polar with reference to an ellipse.

(β) *On the Section of a Cone.*

The treatise *On the Section of a Cone* is even less important, although Serenus claims originality for it. It deals mainly with the areas of triangular sections of right or scalene cones made by planes passing through the vertex and either through the axis or not through the axis, showing when the area of a certain triangle of a particular class is a maximum, under what conditions two triangles of a class may be equal in area, and so on, and solving in some easy cases the problem of finding triangular sections of given area. This sort of investigation occupies Props. 1-57 of the work, these propositions including various lemmas required for the proofs of the substantive theorems. Props. 58-69 constitute a separate section of the book dealing with the volumes of right cones in relation to their heights, their bases and the areas of the triangular sections through the axis.

The essence of the first portion of the book up to Prop. 57 is best shown by means of modern notation. We will call h the height of a right cone, r the radius of the base; in the case of an oblique cone, let p be the perpendicular from the vertex to the plane of the base, d the distance of the foot of this perpendicular from the centre of the base, r the radius of the base.

Consider first the right cone, and let $2x$ be the base of any triangular section through the vertex, while of course $2r$ is the base of the triangular section through the axis. Then, if A be the area of the triangular section with base $2x$,

$$A = x \sqrt{r^2 - x^2 + h^2}.$$

serving that the sum of x^2 and $r^2 - x^2 + h^2$ is constant, we have that A^2 , and therefore A , is a maximum when

$$x^2 = r^2 - x^2 + h^2, \quad \text{or} \quad x^2 = \frac{1}{2}(r^2 + h^2);$$

since x is not greater than r , it follows that, for a real value of x (other than r), h is less than r , or the cone is obtuse-angled. When h is not less than r , the maximum triangle is isosceles through the axis and vice versa (Props. 5, 8); if $h = r$, the maximum triangle is also right-angled (Prop. 13).

The triangle with base $2c$ is equal to the triangle through the axis, $h^2 r^2 = c^2 (r^2 - c^2 + h^2)$, or $(r^2 - c^2)(c^2 - h^2) = 0$, and, if $c < r$, $h = c$, so that $h < r$ (Prop. 10). If x lies between r and c in this case, $(r^2 - x^2)(x^2 - h^2) > 0$ or $x^2(r^2 - x^2 + h^2) > h^2 r^2$, the triangle with base $2x$ is greater than either of the triangles with bases $2r$, $2c$, or $2h$ (Prop. 11).

In the case of the scalene cone Serenus compares individual triangular sections belonging to one of three classes with other sections of the same class as regards their area. The classes

1. axial triangles, including all sections through the axis;

2. isosceles sections, i.e. the sections the bases of which are perpendicular to the projection of the axis of the cone on the plane of the base;

3. a set of triangular sections the bases of which are (α) the diameter of the circular base which passes through the foot of the perpendicular from the vertex to the plane of the base, and (β) the chords of the circular base parallel to that diameter.

After two preliminary propositions (15, 16) and some lemmas, Serenus compares the areas of the first class of triangles through the axis. If, as we said, p is the perpendicular from the vertex to the plane of the base, d the distance from the foot of this perpendicular from the centre of the base, and θ the angle which the base of any axial triangle with area A makes with the base of the axial triangle passing through the perpendicular,

$$A = r\sqrt{(p^2 + d^2 \sin^2 \theta)}.$$

This area is a minimum when $\theta = 0$, and increases with θ

until $\theta = \frac{1}{2}\pi$ when it is a maximum, the triangle being then isosceles (Prop. 24).

In Prop. 29 Serenus takes up the third class of sections with bases parallel to d . If the base of such a section is $2x$,

$$A = x \sqrt{(r^2 - x^2 + p^2)}$$

and, as in the case of the right cone, we must have for a real maximum value

$$x^2 = \frac{1}{2}(r^2 + p^2), \text{ while } x < r,$$

so that, for a real value of x other than r , p must be less than r , and, if p is not less than r , the maximum triangle is that which is perpendicular to the base of the cone and has $2r$ for its base (Prop. 29). If $p < r$, the triangle in question is not the maximum of the set of triangles (Prop. 30).

Coming now to the isosceles sections (2), we may suppose 2θ to be the angle subtended at the centre of the base by the base of the section in the direction away from the projection of the vertex. Then

$$A = r \sin \theta \sqrt{p^2 + (d + r \cos \theta)^2}.$$

If A_0 be the area of the isosceles triangle through the axis, we have

$$\begin{aligned} A_0^2 - A^2 &= r^2(p^2 + d^2) - r^2 \sin^2 \theta (p^2 + d^2 + r^2 \cos^2 \theta + 2dr \cos \theta) \\ &= r^2(p^2 + d^2) \cos^2 \theta - r^4 \sin^2 \theta \cos^2 \theta - 2dr^3 \cos \theta \sin^2 \theta. \end{aligned}$$

If $A = A_0$, we must have for triangles on the side of the centre of the base of the cone towards the vertex of the cone (since $\cos \theta$ is negative for such triangles)

$$p^2 + d^2 < r^2 \sin^2 \theta, \text{ and a fortiori } p^2 + d^2 < r^2 \text{ (Prop. 35).}$$

If $p^2 + d^2 \geq r^2$, A_0 is always greater than A , so that A_0 is the maximum isosceles triangle of the set (Props. 31, 32).

If A is the area of any one of the isosceles triangles with bases on the side of the centre of the base of the cone away from the projection of the vertex, $\cos \theta$ is positive and A_0 is proved to be neither the minimum nor the maximum triangle of this set of triangles (Props. 36, 40-4).

In Prop. 45 Serenus returns to the set of triangular sections through the axis, proving that the feet of the perpendiculars from the vertex of the cone on their bases all lie on a circle the diameter of which is the straight line joining the centre of

the base of the cone to the projection of the vertex on its plane; the areas of the axial triangles are therefore proportional to the generators of the cone with the said circle as base and the same vertex as the original cone. Prop. 50 is to the effect that, if the axis of the cone is equal to the radius of the base, the least axial triangle is a mean proportional between the greatest axial triangle and the isosceles triangular section perpendicular to the base; that is, with the above notation, if $r = \sqrt{(p^2 + d^2)}$, then $r \sqrt{(p^2 + d^2)} : rp = rp : p \sqrt{(r^2 - d^2)}$, which is indeed obvious.

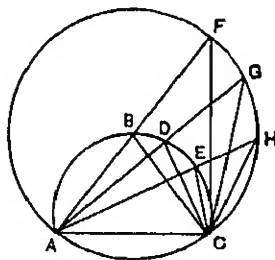
Prop. 57 is interesting because of the lemmas leading to it. It proves that the greater axial triangle in a scalene cone has the greater perimeter, and conversely. This is proved by means of the lemma (Prop. 54), applied to the variable sides of axial triangles, that if $a^2 + d^2 = b^2 + c^2$ and $a > b \geq c > d$, then $a + d < b + c$ (a, d are the sides other than the base of one axial triangle, and b, c those of the other axial triangle compared with it; and if ABC, ADE be two axial triangles and O the centre of the base, $BA^2 + AC^2 = DA^2 + AE^2$ because each of these sums is equal to $2AO^2 + 2BO^2$, Prop. 17). This proposition again depends on the lemma (Props. 52, 53) that, if straight lines be 'inflected' from the ends of the base of a segment of a circle to the curve (i.e. if we join the ends of the base to any point on the curve) the line (i.e. the sum of the chords) is greatest when the point taken is the middle point of the arc, and diminishes as the point is taken farther and farther from that point.

Let B be the middle point of the arc of the segment ABC , D, E any other points on the curve towards C ; I say that

$$AB + BC > AD + DC > AE + EC.$$

With B as centre and BA as radius describe a circle, and produce AB, AD, AE to meet this circle in F, G, H . Join FC, GC, HC .

Since $AB = BC = BF$, we have $AF = AB + BC$. Also the angles BFC, BCF are equal, and each of them is half of the angle ABC .



Again $\angle AGC = \angle AFC = \frac{1}{2}\angle ABC = \frac{1}{2}\angle ADC$;

therefore the angles DGC , DCG are equal and $DG = DC$;

therefore $AG = AD + DC$.

Similarly $EH = EC$ and $AH = AE + EC$.

But, by Eucl. III. 7 or 15, $AF > AG > AH$, and so on;

therefore $AB + BC > AD + DC > AE + EC$, and so on.

In the particular case where the segment ABC is a semi-circle $AB^2 + BC^2 = AC^2 = AD^2 + DC^2$, &c., and the result of Prop. 57 follows.

Props. 58-69 are propositions of this sort: In equal right cones the triangular sections through the axis are reciprocally proportional to their bases and conversely (Props. 58, 59); right cones of equal height have to one another the ratio duplicate of that of their axial triangles (Prop. 62); right cones which are reciprocally proportional to their bases have axial triangles which are to one another reciprocally in the triplicate ratio of their bases and conversely (Props. 66, 67); and so on.

THEON OF ALEXANDRIA lived towards the end of the fourth century A.D. Suidas places him in the reign of Theodosius I (379-95); he tells us himself that he observed a solar eclipse at Alexandria in the year 365, and his notes on the chronological tables of Ptolemy extend down to 372.

Commentary on the *Syntaxis*.

We have already seen him as the author of a commentary on Ptolemy's *Syntaxis* in eleven Books. This commentary is not calculated to give us a very high opinion of Theon's mathematical calibre, but it is valuable for several historical notices that it gives, and we are indebted to it for a useful account of the Greek method of operating with sexagesimal fractions, which is illustrated by examples of multiplication, division, and the extraction of the square root of a non-square number by way of approximation. These illustrations of numerical calculation have already been given above (vol. i,

58-63). Of the historical notices we may mention the following. (1) Theon mentions the treatise of Menelaus *On Chords in a Circle*, i.e. Menelaus's Table of Chords, which came between the similar Tables of Hipparchus and Ptolemy. (2) A citation from Diophantus furnishes incidentally a lower limit to the date of the *Arithmetica*. (3) It is in the commentary on Ptolemy that Theon tells us that the second part of Euclid's Book III relating to *sectors* in equal circles was inserted by him in his edition of the *Elements*, a notice which is of capital importance in that it enables the Theonine manuscripts of Euclid to be distinguished from the ante-Theonine, and is therefore the key to the question how far the genuine text of Euclid was altered in Theon's edition. (4) As we have seen (pp. 207 sq.), Theon, à propos of an allusion of Ptolemy to the theory of isoperimetric figures, has preserved for us several propositions from the treatise by Zenodorus on that subject.

Theon's edition of Euclid's *Elements*.

We are able to judge of the character of Theon's edition of Euclid by a comparison between the Theonine manuscripts and the famous Vatican MS. 190, which contains an earlier text than Theon's, together with certain fragments of other papyri. It appears that, while Theon took some pains to follow older manuscripts, it was not so much his object to get the most authoritative text as to make what he considered improvements of one sort or other. (1) He made emendations where he found, or thought he found, mistakes in the original; while he tried to remove some real blots, he corrected other passages too hastily when a little more consideration would have shown that Euclid's words are right or could be excused, and offer no difficulty to an intelligent reader. He made emendations intended to improve the form or style of Euclid; in general they were prompted by a desire to eliminate anything which was out of the common in expression or in form, in order to reduce the language to one and the same standard or norm. (3) He bestowed, however, most attention upon additions designed to supplement or explain the original; (a) he interpolated whole propositions where he thought them necessary or useful, e.g. the addition to VI. 33

already referred to, a second case to VI. 27, a porism or corollary to II. 4, a second porism to III. 16, the proposition VII. 22, a lemma after X. 12, besides alternative proofs here and there; (b) he added words for the purpose of making smoother and clearer, or more precise, things which Euclid had expressed with unusual brevity, harshness, or carelessness; (c) he supplied intermediate steps where Euclid's argument seemed too difficult to follow. In short, while making only inconsiderable additions to the content of the *Elements*, he endeavoured to remove difficulties that might be felt by learners in studying the book, as a modern editor might do in editing a classical text-book for use in schools; and there is no doubt that his edition was approved by his pupils at Alexandria for whom it was written, as well as by later Greeks, who used it almost exclusively, with the result that the more ancient text is only preserved complete in one manuscript.

Edition of the *Optics* of Euclid.

In addition to the *Elements*, Theon edited the *Optics* of Euclid; Theon's recension as well as the genuine work is included by Heiberg in his edition. It is possible that the *Catoptrica* included by Heiberg in the same volume is also by Theon.

Next to Theon should be mentioned his daughter HYPATIA, who is mentioned by Theon himself as having assisted in the revision of the commentary on Ptolemy. This learned lady is said to have been mistress of the whole of pagan science, especially of philosophy and medicine, and by her eloquence and authority to have attained such influence that Christianity considered itself threatened, and she was put to death by a fanatical mob in March 415. According to Suidas she wrote commentaries on Diophantus, on the Astronomical Canon (of Ptolemy) and on the Conics of Apollonius. These works have not survived, but it has been conjectured (by Tannery) that the remarks of Psellus (eleventh century) at the beginning of his letter about Diophantus, Anatolius, and the Egyptian method of arithmetical reckoning were taken bodily from some manuscript of Diophantus containing an ancient and systematic commentary which may very well have been that of Hypatia. Possibly her commentary may have extended

to the first six Books, in which case the fact that Hypatia wrote a commentary on them may account for the survival of these Books while the rest of the thirteen were first forgotten and then lost.

It will be convenient to take next the series of Neo-Platonist commentators. It does not appear that Ammonius Saccas (about A.D. 175-250), the founder of Neo-Platonism, or his pupil Plotinus (A.D. 204-69), who first expounded the doctrines in systematic form, had any special connexion with mathematics, but PORPHYRY (about 232-304), the disciple of Plotinus and the reviser and editor of his works, appears to have written a commentary on the *Elements*. This we gather from Proclus, who quotes from Porphyry comments on Eucl. I. 14 and 26 and alternative proofs of I. 18, 20. It is possible that Porphyry's work may have been used later by Pappus in writing his own commentary, and Proclus may have got his references from Pappus, but the form of these references suggests that he had direct access to the original commentary of Porphyry.

AMBLICHUS (died about A.D. 330) was the author of a commentary on the *Introductio arithmetica* of Nicomachus, and other works which have already been mentioned. He was a pupil of Porphyry as well as of Anatolius, also a disciple of Porphyry.

But the most important of the Neo-Platonists to the historian of mathematics is PROCLUS (A.D. 410-85). Proclus received his early training at Alexandria, where Olympiodorus was his instructor in the works of Aristotle, and mathematics was taught him by one Heron (of course a different Heron from the 'mechanical Hero' of the *Metrica*, p. 10). He afterwards went to Athens, where he learnt the Neo-Platonic philosophy from Plutarch, the grandson of Nestorius, and from his pupil Syrianus, and became one of its most prominent exponents. He speaks everywhere with the highest respect of his masters, and was in turn regarded with extravagant veneration by his contemporaries, as we learn from Marinus, his pupil and biographer. On the death of Syrianus he was put at the head of the Neo-Platonic school. He was a man of untiring industry, as is shown by the

number of books which he wrote, including a large number of commentaries, mostly on the dialogues of Plato (e.g. the *Timaeus*, the *Republic*, the *Parmenides*, the *Cratylus*). He was an acute dialectician and pre-eminent among his contemporaries in the range of his learning; he was a competent mathematician; he was even a poet. At the same time he was a believer in all sorts of myths and mysteries, and a devout worshipper of divinities both Greek and Oriental. He was much more a philosopher than a mathematician. In his commentary on the *Timaeus*, when referring to the question whether the sun occupies a middle place among the planets, he speaks as no real mathematician could have spoken, rejecting the view of Hipparchus and Ptolemy because *ὁ θεουργός* (sc. the Chaldean, says Zeller) thinks otherwise, 'whom it is not lawful to disbelieve'. Martin observes too, rather neatly, that 'for Proclus the Elements of Euclid had the good fortune not to be contradicted either by the Chaldean Oracles or by the speculations of Pythagoreans old and new'.

Commentary on Euclid, Book I.

For us the most important work of Proclus is his commentary on Euclid, Book I, because it is one of the main sources of our information as to the history of elementary geometry. Its great value arises mainly from the fact that Proclus had access to a number of historical and critical works which are now lost except for fragments preserved by Proclus and others.

(a) Sources of the Commentary.

The historical work the loss of which is most deeply to be deplored is the *History of Geometry* by Eudemus. There appears to be no reason to doubt that the work of Eudemus was accessible to Proclus at first hand. For the later writers Simplicius and Eutocius refer to it in terms such as leave no doubt that *they* had it before them. Simplicius, quoting Eudemus as the best authority on Hippocrates's quadratures of lunes, says he will set out what Eudemus says 'word for word', adding only a little explanation in the shape of references to Euclid's *Elements* 'owing to the memorandum-like style of Eudemus, who sets out his explanations in the abbre-

iated form usual with ancient writers. Now in the second book of the history of geometry he writes as follows'.¹ In the same manner Eutocius speaks of the paralogisms handed down in connexion with the attempts of Hippocrates and Antiphon to square the circle, 'with which I imagine that all persons are accurately acquainted who have examined (ἐπεσκεμμένους) the geometrical history of Eudemus and know the *Ceria Aristotelica*'.²

The references by Proclus to Eudemus by name are not indeed numerous; they are five in number; but on the other hand he gives at least as many other historical data which can with great probability be attributed to Eudemus.

Proclus was even more indebted to Geminus, from whom he borrows long extracts, often mentioning him by name—there are some eighteen such references—but often omitting to do so. We are able to form a tolerably certain judgement as to the origin of the latter class of passages on the strength of the similarity of the subjects treated and the views expressed to those found in the acknowledged extracts. As we have seen, the work of Geminus mainly cited seems to have borne the title *The Doctrine or Theory of the Mathematics*, which was a very comprehensive work dealing, in a portion of it, with the 'classification of mathematics'.

We have already discussed the question of the authorship of the famous historical summary given by Proclus. It is divided, as every one knows, into two distinct parts between which comes the remark, 'Those who compiled histories bring the development of this science up to this point. Not much younger than these is Euclid, who', &c. The ultimate source at any rate of the early part of the summary must presumably have been the great work of Eudemus above mentioned.

It is evident that Proclus had before him the original works of Plato, Aristotle, Archimedes and Plotinus, the *Συμμικτά* of Porphyry and the works of his master Syrianus, as well as a group of works representing the Pythagorean tradition on its mystic, as distinct from its mathematical, side, from Philolaus downwards, and comprising the more or less apocryphal

¹ Simplicius on Arist. *Phys.*, p. 60. 28, Diels.

² Archimedes, ed. Heib., vol. iii, p. 228. 17-19.

τὸς λόγος of Pythagoras, the *Oracles* (λόγια) and Orphic verses.

The following will be a convenient summary of the other works used by Proclus, and will at the same time give an indication of the historical value of his commentary on Euclid, Book I:

Eudemus: *History of Geometry*.

Geminus: *The Theory of the Mathematical Sciences*.

Heron: *Commentary on the Elements of Euclid*.

Porphry: " " "

Pappus: " " "

Apollonius of Perga: A work relating to elementary geometry.

Ptolemy: *On the parallel-postulate*.

Posidonius: A book controverting Zeno of Sidon.

Carpus: *Astronomy*.

Syrianus: A discussion on the *angle*.

(β) *Character of the Commentary.*

We know that in the Neo-Platonic school the pupils learnt mathematics; and it is clear that Proclus taught this subject, and that this was the origin of his commentary. Many passages show him as a master speaking to scholars; in one place he speaks of 'my hearers'.¹ Further, the pupils whom he was addressing were *beginners* in mathematics; thus in one passage he says that he omits 'for the present' to speak of the discoveries of those who employed the curves of Nicomedes and Hippias for trisecting an angle, and of those who used the Archimedean spiral for dividing an angle in a given ratio, because these things would be 'too difficult for beginners'.² But there are signs that the commentary was revised and re-edited for a larger public; he speaks for instance in one place of 'those who will come across his work'.³ There are also passages, e.g. passages about the cylindrical helix, conchoids and cissoids, which would not have been understood by the beginners to whom he lectured.

¹ Proclus on Eucl. I, p. 210. 19.

² *Ib.*, p. 272. 12.

³ *Ib.*, p. 84. 9.

commentary opens with two Prologues. The first is on mathematics in general and its relation to, and use in, philosophy, from which Proclus passes to the classification of mathematics. Prologue II deals with geometry generally and subject-matter according to Plato, Aristotle and others. In this section comes the famous summary (pp. 64-8) beginning with a eulogium of Euclid, with particular reference to the admirable discretion shown in the selection of the propositions which should constitute the *Elements* of geometry, the ordering of the whole subject-matter, the exactness and conclusiveness of the demonstrations, and the power with which every question is handled. Generalities follow, such as a discussion of the nature of *elements*, the distinction between theorems and problems according to different authorities, and by a division of Book I into three main sections, (1) the construction and properties of triangles and their parts and comparison between triangles in respect of their angles and sides, (2) the properties of parallels and parallelograms, their construction from certain data, and (3) the bringing of angles and parallelograms into relation as regards area. Coming to the Book itself, Proclus deals historically and logically with all the definitions, postulates and axioms in Book I. The notes on the postulates and axioms are preceded by a general discussion of the principles of geometry, hypotheses, postulates and axioms, and their relation to one another; here as usual Proclus quotes the opinions of all the important authorities. Again, when he comes to Prop. 1, he discusses once more the difference between theorems and problems, then sets out and explains the formal divisions of a proposition, the *enunciation* (πρότασις), the *setting-out* (ὑπόθεσις), the *definition* or *specification* (διορισμός), the *construction* (κατασκευή), the *proof* (ἀπόδειξις), the *conclusion* (ἐπάραμα), and finally a number of other technical terms, things said to be *given*, in the various senses of this term, *lemma*, the *case*, the *porism* in its two senses, the *objection* (ἀντίρρησης), the *reduction* of a problem, *reductio ad absurdum*, *analysis* and *synthesis*.

In his comments on the separate propositions Proclus usually proceeds in this way: first he gives explanations of Euclid's proofs, secondly he gives a few different

cases, mainly for the sake of practice, and thirdly he addresses himself to refuting objections which cavillers had taken or might take to particular propositions or arguments. He does not seem to have had any notion of correcting or improving Euclid; only in one place does he propose anything of his own to get over a difficulty which he finds in Euclid; this is where he tries to prove the parallel-postulate, after giving Ptolemy's attempt to prove it and pointing out objections to Ptolemy's proof.

The book is evidently almost entirely a compilation, though a compilation 'in the better sense of the term'. The *onus probandi* is on any one who shall assert that anything in it is Proclus's own; very few things can with certainty be said to be so. Instances are (1) remarks on certain things which he quotes from Pappus, since Pappus was the last of the commentators whose works he seems to have used, (2) a defence of Geminus against Carpus, who criticized Geminus's view of the difference between theorems and problems, and perhaps (3) criticisms of certain attempts by Apollonius to improve on Euclid's proofs and constructions; but the only substantial example is (4) the attempted proof of the parallel-postulate, based on an 'axiom' to the effect that, 'if from one point two straight lines forming an angle be produced *ad infinitum*, the distance between them when so produced *ad infinitum* exceeds any finite magnitude (i. e. length)', an assumption which purports to be the equivalent of a statement in Aristotle.¹ Philoponus says that Proclus as well as Ptolemy wrote a whole book on the parallel-postulate.²

It is still not quite certain whether Proclus continued his commentaries beyond Book I. He certainly intended to do so, for, speaking of the trisection of an angle by means of certain curves, he says, 'we may perhaps more appropriately examine these things on the third Book, where the writer of the Elements bisects a given circumference', and again, after saying that of all parallelograms which have the same perimeter the square is the greatest 'and the rhomboid least of all', he adds, 'But this we will prove in another place, for it is more appropriate to the discussion of the hypotheses of the

¹ *De caelo*, i. 5, 271 b 28-30.

² Philoponus on *Anal. Post.* i. 10, p. 214 n 9-12, Brandis.

second Book'. But at the time when the commentary on Book I was written he was evidently uncertain whether he could be able to continue it, for at the end he says, 'For my part, if I should be able to discuss the other Books in the same way, I should give thanks to the gods; but, if other cares should draw me away, I beg those who are attracted by this subject to complete the exposition of the other Books as well, following the same method and addressing themselves throughout to the deeper and more sharply defined questions involved'.¹ Wachsmuth, finding a Vatican manuscript containing a collection of scholia on Books I, II, V, VI, X, headed *Εἰς τὰ ὑκλείδου στοιχεῖα προλαμβανόμενα ἐκ τῶν Πρόκλου σποράδην καὶ κατ' ἐπιτομήν*, and seeing that the scholia on Book I were extracts from the extant commentary of Proclus, concluded that those on the other Books were also from Proclus; but the *προ-* in *προλαμβανόμενα* rather suggests that only the scholia to Book I are from Proclus. Heiberg found and published in 1903 a scholium to X. 9, in which Proclus is expressly quoted as the authority, but he does not regard this circumstance as conclusive. On the other hand, Heiberg has noted two facts which go against the view that Proclus wrote on the later Books: (1) the scholiast's copy of Proclus was not much better than our manuscripts; in particular, it had the same lacunae in the notes to I. 36, 37, and I. 41-3; this makes it improbable that the scholiast had further commentaries of Proclus which have vanished from us; (2) there is no trace in the scholia of the notes which Proclus promised in the passages already referred to. All, therefore, that we can say is that, while the Wachsmuth scholia *may* be extracts from Proclus, it is on the whole improbable.

Hypotyposis of Astronomical Hypotheses.

Another extant work of Proclus which should be referred to is his *Hypotyposis of Astronomical Hypotheses*, a sort of readable and easy introduction to the astronomical system of Hipparchus and Ptolemy. It has been well edited by Manitius (Teubner, 1909). Three things may be noted as

¹ Proclus on Eucl. I, p. 492. 9-15.

regards this work. It contains¹ a description of the method of measuring the sun's apparent diameter by means of Heron's water-clock, which, by comparison with the corresponding description in Theon's commentary to the *Syntaxis* of Ptolemy, is seen to have a common source with it. That source is Pappus, and, inasmuch as Proclus has a figure (reproduced by Manitius in his text from one set of manuscripts) corresponding to the description, while the text of Theon has no figure, it is clear that Proclus drew directly on Pappus, who doubtless gave, in his account of the procedure, a figure taken from Heron's own work on water-clocks. A simple proof of the equivalence of the epicycle and eccentric hypotheses is quoted by Proclus from one Hilarius of Antioch.² An interesting passage is that in chap. 4 (p. 130, 18) where Sosigenes the Peripatetic is said to have recorded in his work 'on reacting spheres' that an *annular* eclipse of the sun is sometimes observed at times of perigee; this is, so far as I know, the only allusion in ancient times to annular eclipses, and Proclus himself questions the correctness of Sosigenes's statement.

Commentary on the *Republic*.

The commentary of Proclus on the *Republic* contains some passages of great interest to the historian of mathematics. The most important is that³ in which Proclus indicates that Props. 9, 10 of Euclid, Book II, are Pythagorean propositions invented for the purpose of proving geometrically the fundamental property of the series of 'side-' and 'diameter-' numbers, giving successive approximations to the value of $\sqrt{2}$ (see vol. i, p. 93). The explanation⁴ of the passage in Plato about the Geometrical Number is defective and disappointing, but it contains an interesting reference to one Paterius, of date presumably intermediate between Nestorius and Proclus. Paterius is said to have made a calculation, in units and submultiples, of the lengths of different segments of

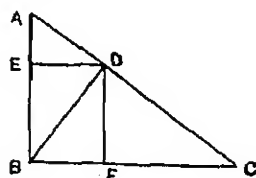
¹ Proclus, *Hypotyposis*, c. 4, pp. 120-22.

² *Ib.*, c. 3, pp. 76, 17 sq.

³ *Procli Diadochi in Platonis Rempublicam Commentarii*, ed. Kroll, vol. ii, p. 27.

⁴ *Ib.*, vol. ii, pp. 36-42.

ght lines in a figure formed by taking a triangle with
s 3, 4, 5 as ABC , then drawing
from the right angle B perpen-
lar to AC , and lastly drawing
pendiculars DE , DF to AB , BC .
Diagram in the text with the
ths of the segments shown along-
them in the usual numerical
ation shows that Paterius obtained from the data $AB = 3$,
 $BC = 4$, $CA = 5$ the following:



$$DC = \gamma \epsilon' = 3\frac{1}{5}$$

$$BD = \beta \gamma' \epsilon' = 2\frac{1}{3} \cdot \frac{1}{15} [= 2\frac{2}{5}]$$

$$AD = \alpha \delta' \kappa' = 1\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{20} [= 1\frac{1}{5}]$$

$$BC = \beta \delta' \kappa' \rho' = 2\frac{1}{2} \cdot \frac{1}{20} \cdot \frac{1}{100} [= 2\frac{1}{25}]$$

$$EB = \alpha \gamma' \epsilon' \kappa' = 1\frac{1}{3} \cdot \frac{1}{15} \cdot \frac{1}{25} [= 1\frac{1}{25}]$$

$$BE = \alpha \delta' \gamma' \epsilon' \nu' = 1\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{15} \cdot \frac{1}{50} [= 1\frac{2}{25}]$$

$$EA = \alpha \epsilon' \rho' \epsilon' = 1\frac{1}{5} \cdot \frac{1}{75} [= 1\frac{2}{25}]$$

s is an example of the Egyptian method of stating frac-
as preceding by some three or four centuries the exposition
the same method in the papyrus of Akhmim.

MARINUS of Neapolis, the pupil and biographer of Proclus,
ote a commentary or rather introduction to the *Data* of
Euclid.¹ It is mainly taken up with a discussion of the
question τί τὸ δεδομένον, what is meant by *given*? There
are apparently many different definitions of the term *given*
in earlier and later authorities. Of those who tried to define
it in the simplest way by means of a single *differentia*, three
are mentioned by name. Apollonius in his work on *νεύσεις*
and his 'general treatise' (presumably that on elementary
geometry) described the *given* as *assigned* or *fixed* (τεταγ-
μένον), Diodorus called it *known* (γνώριμον); others regarded
it as *rational* (ρήτ론) and Ptolemy is classed with these, rather
oddly, because 'he called those things given the measure of
which is given either exactly or approximately'. Others

¹ See Heiberg and Menge's *Euclid*, vol. vi, pp. 234-56.

combined two of these ideas and called it *assigned* or *fixed* and *procurable* or capable of being found (*πόριμον*); others 'fixed and known', and a third class 'known and procurable'. These various views are then discussed at length.

DOMNINUS of Larissa, a pupil of Syrianus at the same time as Proclus, wrote a *Manual of Introductory Arithmetic* *ἐγχειρίδιον ἀριθμητικῆς εἰσαγωγῆς*, which was edited by Boissonade¹ and is the subject of two articles by Tannery,² who also left a translation of it, with prolegomena, which has since been published.³ It is a sketch of the elements of the theory of numbers, very concise and well arranged, and is interesting because it indicates a serious attempt at a reaction against the *Introductio arithmetica* of Nicomachus and a return to the doctrine of Euclid. Besides Euclid, Nicomachus and Theon of Smyrna, Domninus seems to have used another source, now lost, which was also drawn upon by Iamblichus. At the end of this work Domninus foreshadows a more complete treatise on the theory of numbers under the title *Elements of Arithmetic* (*ἀριθμητικὴ στοιχείωσις*), but whether this was ever written or not we do not know. Another tract attributed to Domninus *πῶς ἔστι λόγον ἐκ λόγου ἀφελεῖν* (how a ratio can be taken out of a ratio) has been published with a translation by Ruelle⁴; if it is not by Domninus, it probably belongs to the same period.

A most honourable place in our history must be reserved for SIMPLICIUS, who has been rightly called 'the excellent Simplicius, the Aristotle-commentator, to whom the world can never be grateful enough for the preservation of the fragments of Parmenides, Empedocles, Anaxagoras, Melissus, Theophrastus and others' (v. Wilamowitz-Möllendorff). He lived in the first half of the sixth century and was a pupil, first of Ammonius of Alexandria, and then of Damascius, the last head of the Platonic school at Athens. When in the year 529 the Emperor Justinian, in his zeal to eradicate paganism, issued an edict forbidding the teaching of philo-

¹ *Anecdota Graeca*, vol. iv, pp. 413-29.

² *Mémoires scientifiques*, vol. ii, nos. 85, 40.

³ *Revue des études grecques*, 1906, pp. 359-82; *Mémoires scientifiques*, vol. iii, pp. 256-81.

⁴ *Revue de Philologie*, 1883, p. 83 sq.

at Athens, the last members of the school, including Simplicius and Simplicius, migrated to Persia, but returned 533 to Athens, where Simplicius continued to teach for time though the school remained closed.

Extracts from Eudemus.

Simplicius we owe two long extracts of capital importance for the history of mathematics and astronomy. The first is his account, based upon and to a large extent quoted verbatim from Eudemus's *History of Geometry*, of the attempt by Hippocrates of Chios to square the circle and of the quadratures of lunes by Hippocrates of Chios. It is contained in Simplicius's commentary on Aristotle's *Physics*,¹ and has been the subject of considerable literature extending from 1870, the date when Brötschneider first called attention to it, to the latest edition with translation and notes by Rudolph (Leubner, 1907).

It has already been discussed (vol. i, pp. 183-99). The second, and not less important, of the two passages is that containing the elaborate and detailed account of the system of concentric spheres, as first invented by Eudoxus for explaining the apparent motion of the sun, moon, and planets, and of the modifications made by Callippus and Aristotle. It is contained in the commentary on Aristotle's *De caelo*;² Simplicius quotes largely from Sosigenes the Peripatetic (second century A.D.), observing that he in his turn drew from Eudemus, who dealt with the subject in the second volume of his *History of Astronomy*. It is this passage of Simplicius which, along with a passage in Aristotle's *Metaphysics*,³ enabled Schiaparelli to reconstruct Eudoxus's system (vol. i, pp. 329-34). Nor must it be forgotten that it is in Simplicius's commentary on the *Physics*⁴ that the extract from Geminus's summary of the *Meteorologica* of Posidonius is given, which was used by Schiaparelli to support his view that it was Heraclides of Pontus, not Aristarchus of Samos, who first propounded the heliocentric hypothesis.

Simplicius also wrote a commentary on Euclid's *Elements*, I, from which an-Nairizi, the Arabian commentator,

¹ Simplicius, *in Phys.*, pp. 54-69, ed. Diels.

² Simplicius, *on Arist. De caelo*, p. 488. 18-24 and pp. 493-500, ed. Heiberg.

³ *Metaph.* A. 8, 1073 b 17-1074 n 14.

⁴ Simplicius, *in Phys.*, pp. 291-2, ed. Diels.

made valuable extracts, including the account of the attempt of 'Aganis' to prove the parallel-postulate (see pp. 228-30 above).

Contemporary with Simplicius, or somewhat earlier, was Eutocius, the commentator on Archimedes and Apollonius. As he dedicated the commentary on Book I *On the Sphere and Cylinder* to Aminonius (a pupil of Proclus and teacher of Simplicius), who can hardly have been alive after A.D. 510, Eutocius was probably born about A.D. 480. His date used to be put some fifty years later because, at the end of the commentaries on Book II *On the Sphere and Cylinder* and on the *Measurement of a Circle*, there is a note to the effect that 'the edition was revised by Isidorus of Miletus, the mechanical engineer, *our teacher*'. But, in view of the relation to Ammonius, it is impossible that Eutocius can have been a pupil of Isidorus, who was younger than Anthemius of Tralles, the architect of Saint Sophia at Constantinople in 532, whose work was continued by Isidorus after Anthemius's death about A.D. 534. Moreover, it was to Anthemius that Eutocius dedicated, separately, the commentaries on the first four Books of Apollonius's *Conics*, addressing Anthemius as 'my dear friend'. Hence we conclude that Eutocius was an elder contemporary of Anthemius, and that the reference to Isidorus is by an editor of Eutocius's commentaries who was a pupil of Isidorus. For a like reason, the reference in the commentary on Book II *On the Sphere and Cylinder*¹ to a διαβήτης invented by Isidorus 'our teacher' for drawing a parabola must be considered to be an interpolation by the same editor.

Eutocius's commentaries on Archimedes apparently extended only to the three works, *On the Sphere and Cylinder*, *Measurement of a Circle* and *Plane Equilibriums*, and those on the *Conics* of Apollonius to the first four Books only. We are indebted to these commentaries for many valuable historical notes. Those deserving special mention here are (1) the account of the solutions of the problem of the duplication of the cube, or the finding of two mean proportionals, by 'Plato', Heron, Philon, Apollonius, Diocles, Pappus, Sporus, Menaechmus, Archytas, Eratosthenes, Nicomedes, (2) the fragment discovered by Eutocius himself containing the

¹ Archimedes, ed. Heiberg, vol. iii, p. 84. 8-11.

ing solution, promised by Archimedes in *On the Sphere and Cylinder*, II. 4, of the auxiliary problem amounting to the solution by means of conics of the cubic equation $x^3 = bx^2 + cx$, (2) the solutions (α) by Dioctes of the original problem of II. 4 without bringing in the cubic, (β) by Dionysius of the auxiliary cubic equation.

ANTHEMIUS of Tralles, the architect, mentioned above, was himself an able mathematician, as is seen from a fragment of the work of his, *On Burning-mirrors*. This is a document of considerable importance for the history of conic sections. Originally edited by L. Dupuy in 1777, it was reprinted in Hermann's *Παραδοξογράφοι* (*Scriptores rerum mirabilium ceteri*), 1839, pp. 149-58. The first and third portions of the fragment are those which interest us.¹ The first gives the solution of the problem, 'To contrive that a ray of the sun (admitted through a small hole or window) shall fall in a fixed spot, without moving away at any hour and season. This is contrived by constructing an elliptical mirror one focus of which is at the point where the ray of the sun is admitted and the other is at the point to which the ray is required to be reflected at all times. Let B be the hole, A the point to which reflection must always take place, BA being in the meridian and parallel to the horizon. Let BC be at right angles to BA , so that CB is an equinoctial ray; and let BD be a ray at the summer solstice, BE a winter ray.

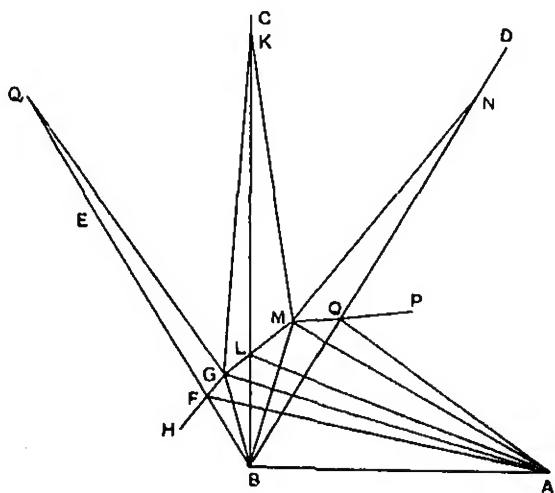
Take F at a convenient distance on BE and measure FQ equal to FA . Draw HFG through F bisecting the angle QFE , and let BG be the straight line bisecting the angle EBC between the winter and the equinoctial rays. Then clearly, FG bisects the angle QFA , if we have a plane mirror in position HFG , the ray BFE entering at B will be reflected

to get the equinoctial ray similarly reflected to A , join GA , with G as centre and GA as radius draw a circle meeting BC in K . Bisect the angle KGA by the straight line GLM bisecting BK in L and terminated at M , a point on the bisector of the angle CBD . Then LM bisects the angle KLA also, and $KL = LA$, and $KM = MA$. If then GLM is a plane mirror, a ray BL will be reflected to A .

¹ See *Bibliotheca mathematica*, vii, 1907, pp. 225-33.

By taking the point N on BD such that $MN = MA$, and bisecting the angle NMA by the straight line MOP meeting BD in O , we find that, if MOP is a plane mirror, the ray BO is reflected to A .

Similarly, by continually bisecting angles and making more mirrors, we can get any number of other points of impact. Making the mirrors so short as to form a continuous curve, we get the curve containing all points such that the sum of the distances of each of them from A and B is constant and equal to BQ , BK , or BN . 'If then', says Anthemius, 'we stretch a string passed



round the points A , B , and through the first point taken on the rays which are to be reflected, the said curve will be described, which is part of the so-called "ellipse", with reference to which (i.e. by the revolution of which round BA) the surface of impact of the said mirror has to be constructed.'

We have here apparently the first mention of the construction of an ellipse by means of a string stretched tight round the foci. Anthemius's construction depends upon two propositions proved by Apollonius (1) that the sum of the focal distances of any point on the ellipse is constant, (2) that the focal distances of any point make equal angles with the tangent at that point, and also (3) upon a proposition not found in Apollonius, namely that the straight line joining

the focus to the intersection of two tangents bisects the angle between the straight lines joining the focus to the two points of contact respectively.

In the third portion of the fragment Anthemius proves that parallel rays can be reflected to one single point from a parabolic mirror of which the point is the focus. The *directrix* is used in the construction, which follows, *mutatis mutandis*, the same course as the above construction in the case of the ellipse. As to the supposition of Heiberg that Anthemius may also be the author of the *Fragmentum mathematicum Bobiense*, see above (p. 203).

The Papyrus of Akhmim.

Next in chronological order must apparently be placed the Papyrus of Akhmim, a manual of calculation written in Greek, which was found in the metropolis of Akhmim, the ancient Panopolis, and is now in the Musée du Louvre. It was edited by J. Baillet¹ in 1892. According to the editor, it was written between the sixth and eighth centuries by a Christian. It is interesting because it preserves the Egyptian method of reckoning, with proper fractions written as the sum of primary fractions or submultiples, a method which survived alongside the Greek and was employed, and even exclusively taught, in the East. The advantage of this papyrus, as compared with Ahmos's, is that we can gather the formulæ used for the decomposition of ordinary proper fractions into sums of submultiples. The formulæ for decomposing a proper fraction into the sum of two submultiples may be shown thus:

$$(1) \frac{a}{bc} = \frac{1}{c \cdot \frac{b+c}{a}} + \frac{1}{b \cdot \frac{b+c}{a}}.$$

$$\text{Examples } \frac{2}{11} = \frac{1}{6} \frac{1}{6}, \quad \frac{3}{110} = \frac{1}{70} \frac{1}{77}, \quad \frac{18}{323} = \frac{1}{34} \frac{1}{98}.$$

$$(2) \frac{a}{bc} = \frac{1}{c \cdot \frac{b+mc}{a}} + \frac{1}{b \cdot \frac{b+mc}{a} \cdot \frac{1}{m}}.$$

¹ *Mémoires publiés par les membres de la Mission archéologique française au Caire*, vol. ix, part 1, pp. 1-89.

$$\text{Ex. } \frac{7}{176} = \frac{1}{11 \left(\frac{16+3 \cdot 11}{7} \right)} + \frac{1}{16 \left(\frac{16+3 \cdot 11}{7} \right) \frac{1}{3}} = \frac{1}{77} + \frac{3}{112};$$

$$\text{and again } \frac{3}{112} = \frac{1}{7 \left(\frac{16+2 \cdot 7}{3} \right)} + \frac{1}{16 \left(\frac{16+2 \cdot 7}{3} \right) \frac{1}{2}} = \frac{1}{70} + \frac{1}{80}$$

$$(3) \quad \frac{a}{cdf} = -\frac{1}{c \cdot \frac{cd+df}{a}} + \frac{1}{f \cdot \frac{cd+df}{a}}.$$

Example.

$$\frac{28}{1320} = \frac{28}{10 \cdot 12 \cdot 11} = \frac{1}{10 \cdot \frac{120+132}{28}} + \frac{1}{11 \cdot \frac{120+132}{28}} = \frac{1}{90} + \frac{1}{99}.$$

The object is, of course, to choose the factors of the denominator, and the multiplier m in (2), in such a way as to make the two denominators on the right-hand side integral.

When the fraction has to be decomposed into a sum of three or more submultiples, we take out an obvious submultiple first, then if necessary a second, until one of the formulae will separate what remains into two submultiples. Or we take out a part which is not a submultiple but which can be divided into two submultiples by one of the formulae.

For example, to decompose $\frac{3}{818}$. The factors of 616 are 8.77 or 7.88. Take out $\frac{1}{88}$, and $\frac{3}{818} = \frac{1}{88} \frac{24}{818} = \frac{1}{88} \frac{7}{77} = \frac{1}{88} \frac{1}{77} \frac{2}{77}$; and $\frac{2}{77} = \frac{1}{33} \frac{1}{88}$ by formula (1), so that $\frac{3}{818} = \frac{1}{88} \frac{1}{77} \frac{1}{88} \frac{1}{88}$.

Take $\frac{239}{8480}$. The factors of 6460 are 85.76 or 95.68. Take out $\frac{1}{88}$, and $\frac{239}{8480} = \frac{1}{88} \frac{163}{8480}$. Again take out $\frac{1}{88}$, and we have $\frac{1}{88} \frac{1}{88} \frac{163}{8480}$ or $\frac{1}{88} \frac{1}{88} \frac{1}{88}$. The actual problem here is to find $\frac{1}{323}$ rd of $11\frac{1}{2} \frac{1}{3} \frac{1}{16} \frac{1}{80}$, which latter expression reduces to $\frac{1}{20} \cdot 239$.

The sort of problems solved in the book are (1) the division of a number into parts in the proportion of certain given numbers, (2) the solution of simple equations such as this: From a certain treasure we take away $\frac{1}{13}$ th, then from the remainder $\frac{1}{17}$ th of that remainder, and we find 150 units left; what was the treasure? $\left[\left\{ x - \frac{1}{n}x - \frac{1}{b} \left(x - \frac{1}{n}x \right) - \dots \right\} = R. \right]$

extractions such as: From $\frac{2}{3}$ subtract $\frac{1}{10} \frac{1}{11} \frac{1}{20} \frac{1}{22} \frac{1}{30} \frac{1}{33}$
 $\frac{1}{36} \frac{1}{38} \frac{1}{40} \frac{1}{44} \frac{1}{48} \frac{1}{55} \frac{1}{60} \frac{1}{66} \frac{1}{70} \frac{1}{77} \frac{1}{88} \frac{1}{90} \frac{1}{99} \frac{1}{100} \frac{1}{110}$. Answer, $\frac{1}{10} \frac{1}{36}$.

The book ends with long tables of results obtained (1) by multiplying successive numbers, tens, hundreds and thousands 10,000 by $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, &c., up to $\frac{1}{10}$, (2) by multiplying successive numbers 1, 2, 3 ... n by $\frac{1}{n}$, where n is successive 11, 12, ... and 20; the results are all arranged as the products of integers and submultiples.

Geodæsia of a Byzantine author formerly called, with authority, 'Heron the Younger' was translated into Latin by Barocius in 1572, and the Greek text was published in 1814. A French translation by Vincent.¹ The place of the author's observations was the hippodrome at Constantinople, the date apparently about 938. The treatise was modelled on the *Geometrica* of Alexandria, especially the *Dioptra*, while some measurements of areas and volumes are taken from the *Geometrica*.

MICHAEL PSELLUS lived in the latter part of the eleventh century, since his latest work bears the date 1092. Though called 'first of philosophers', it cannot be said that any survives of his mathematics suits this title. Xylander printed in 1556 the Greek text, with a Latin translation, of a book purporting to be by Psellus on the four mathematical sciences, arithmetic, music, geometry and astronomy, but it is evident that it cannot be entirely Psellus's own work, since the astronomical portion is dated 1008. The arithmetic contains no more than the names and classification of numbers and ratios. The geometry has the extraordinary remark that, 'The opinions differed as to how to find the area of a circle, the method which found most favour was to take the area as the geometric mean between the inscribed and circumscribed polygons; this gives $\pi = \sqrt{8} = 2.82842711$ '. The only thing of value which has any value for us is the letter published by Xylander in his edition of Diophantus.² In this letter Psellus states that both Diophantus and Anatolius (Bishop of Laodicea A.D. 280) wrote on the Egyptian method of reckoning,

¹ *Notices et extraits*, xix, pt. 2, Paris, 1858.

² Diophantus, vol. ii, pp. 87-42.

and that Anatolius's account, which was different and more succinct, was dedicated to Diophantus (this enables us to determine Diophantus's date approximately). He also notes the difference between the Diophantine and Egyptian names for the successive powers of ἀριθμός: the next power after the fourth (δυναμοδύναμις = x^4), i.e. x^5 , the Egyptians called 'the first undescribed' (ἄλογος πρῶτος) or the 'fifth number'; the sixth, x^6 , they apparently (like Diophantus) called the cube-cube; but with them the seventh, x^7 , was the 'second undescribed' or the 'seventh number', the eighth (x^8) was the 'quadruple square' (τετραπλῇ δύναμις), the ninth (x^9) the 'extended cube' (κύβος ἐξελικτός). Tannery conjectures that all these remarks were taken direct from an old commentary on Diophantus now lost, probably Hypatia's.

GEORGIUS PACHYMERES (1242-1310) was the author of a work on the Quadrivium (Σύνταγμα τῶν τεσσάρων μαθημάτων or Τετράβιβλον). The arithmetical portion contains, besides excerpts from Nicomachus and Euclid, a paraphrase of Diophantus, Book I, which Tannery published in his edition of Diophantus¹; the musical section with part of the preface was published by Vincent,² and some fragments from Book IV by Martin in his edition of the *Astronomy* of Theon of Smyrna.

MAXIMUS PLANODES, a monk from Nicomedia, was the envoy of the Emperor Andronicus II at Venice in the year 1297, and lived probably from about 1260 to 1310. He wrote scholia on the first two Books of Diophantus, which are extant and are included in Tannery's edition of Diophantus.³ They contain nothing of particular interest except a number of conspectuses of the working-out of problems of Diophantus written in Diophantus's own notation but with steps in separate lines, and with abbreviations on the left of words indicating the operations (e.g. ἐκθ. = ἐκθεσις, τετρ. = τετραγωνισμός, σύνθ. = σύνθεσις, &c.); the result is to make the work almost as easy to follow as it is in our notation.

Another work of Planudes is called *Ψηφοφολία κατ' Ἰνδούς*, or *Arithmetic after the Indian method*, and was edited as *Das*

¹ Diophantus, vol. ii, pp. 78-122.

² *Notices et extraits*, xvii, 1858, pp. 362-533.

³ Diophantus, vol. ii, pp. 125-255.

Rechenbuch des Maximus Planudes in Greek by Gerhardt (Halle, 1865) and in a German translation by H. Waeschke (Halle, 1878). There was, however, an earlier book under the similar title *Ἀρχὴ τῆς μεγάλης καὶ Ἰνδικῆς ψηφιοφίας* (*sic*), written in 1252, which is extant in the Paris MS. Suppl. Gr. 7; and Planudes seems to have raided this work. He begins with an account of the symbols which, he says, were

invented by certain distinguished astronomers for the most convenient and accurate expression of numbers. There are nine of these symbols (our 1, 2, 3, 4, 5, 6, 7, 8, 9), to which is added another called *Tzifra* (cypher), written 0 and denoting zero. The nine signs as well as this one are Indian.'

But this is, of course, not the first occurrence of the Indian numerals; they were known, except the zero, to Gerbert (Pope Sylvester II) in the tenth century, and were used by Leonardo of Pisa in his *Liber abaci* (written in 1202 and revised in 1228). Planudes used the Persian form of the numerals, differing in this from the writer of the treatise of 1252 referred to, who used the form then current in Italy. It scarcely belongs to Greek mathematics to give an account of Planudes's methods of subtraction, multiplication, &c.

Extraction of the square root.

As regards the extraction of the square root, he claims to have invented a method different from the Indian method and from that of Theon. It does not appear, however, that there was anything new about it. Let us try to see in what the supposed new method consisted.

Planudes describes fully the method of extracting the square root of a number with several digits, a method which is essentially the same as ours. 'This appears to be what he refers to later on as 'the Indian method'. Then he tells us how to find a first approximation to the root when the number is not a complete square.

'Take the square root of the next lower actual square number, and double it: then, from the number the square root of which is required, subtract the next lower square number found, and to the remainder (as numerator) give as denominator the double of the square root already found.'

The example given is $\sqrt{18}$. Since $4^2 = 16$ is the next lower square, the approximate square root is $4 + \frac{2}{2 \cdot 4}$ or $4\frac{1}{4}$.

The formula used is, therefore, $\sqrt{a^2 + b} = a + \frac{b}{2a}$ approximately. (An example in larger numbers is

$$\sqrt{1690196789} = 41112 + \frac{245}{2 \cdot 41112} \text{ approximately.})$$

Planudes multiplies $4\frac{1}{4}$ by itself and obtains $18\frac{1}{16}$, which shows that the value $4\frac{1}{4}$ is not accurate. He adds that he will explain later a method which is more exact and nearer the truth, a method 'which I claim as a discovery made by me with the help of God'. Then, coming to the method which he claims to have discovered, Planudes applies it to $\sqrt{6}$. The object is to develop this in units and sexagesimal fractions. Planudes begins by multiplying the 6 by 3600, making 21600 second-sixtieths, and finds the square root of 21600 to lie between 146 and 147. Writing the 146' as 2 26', he proceeds to find the rest of the approximate square root (2 26' 58" 9'') by the same procedure as that used by Theon in extracting the square root of 4500 and 2 28' respectively. The difference is that in neither of the latter cases does Theon multiply by 3600 so as to reduce the units to second-sixtieths, but he begins by taking the approximate square root of 2, viz. 1, just as he does that of 4500 (viz. 67). It is, then, the multiplication by 3600, or the reduction to second-sixtieths to start with, that constitutes the difference from Theon's method, and this must therefore be what Planudes takes credit for as a new discovery. In such a case as $\sqrt{(2 \ 28')}$ or $\sqrt{3}$, Theon's method has the inconvenience that the number of *minutes* in the second term (34' in the one case and 43' in the other) cannot be found without some trouble, a difficulty which is avoided by Planudes's expedient. Therefore the method of Planudes had its advantage in such a case. But the discovery was not new. For it will be remembered that Ptolemy (and doubtless Hipparchus before him) expressed the chord in a circle subtending an angle of 120° at the centre (in terms of 120th parts of the diameter) as $103^\circ 55' 23''$, which indicates that the first step in calculating $\sqrt{3}$ was to multiply it by 3600, making 10800, the nearest square below which is $103^2 (= 10609)$. In

scholia to Eucl., Book X, the same method is applied. Examples have been given above (vol. i, p. 63). The supposed new method was therefore not only already known to the scholiast, but goes back, in all probability, to Hipparchus.

Two problems.

Two problems given at the end of the Manual of Planudes are worth mention. The first is stated thus: 'A certain man dying himself at the point of death had his desk or safe brought to him and divided his money among his sons with the following words, "I wish to divide my money equally between my sons: the first shall have one piece and $\frac{1}{n}$ th of the rest, the second 2 and $\frac{1}{n}$ th of the remainder, the third 3 and $\frac{1}{n}$ th of the remainder." At this point the father died without getting to the end either of his money or the enumeration of his sons. I wish to know how many sons he had and how much money.' The solution is given as $(n-1)^2$ for the number of coins to be divided and $(n-1)$ for the number of his sons; rather this is how it might be stated, for Planudes takes $n = 7$ arbitrarily. Comparing the shares of the first two we must clearly have

$$1 + \frac{1}{n}(x-1) = 2 + \frac{1}{n}\{x - (1 + \frac{x-1}{n} + 2)\},$$

which gives $x = (n-1)^2$; therefore each of $(n-1)$ sons received $(n-1)$.

The other problem is one which we have already met with, that of finding two rectangles of equal perimeter such that the area of one of them is a given multiple of the area of the other. If n is the given multiple, the rectangles are (n^2-1, n^2-n^2) and $(n-1, n^2-n)$ respectively. Planudes states the solution correctly, but how he obtained it is not clear. We find also in the Manual of Planudes the 'proof by nine' (i.e. by casting out nines), with a statement that it was discovered by the Indians and transmitted to us through the Arabs.

MANUEL MOSCHOPOULOS, a pupil and friend of Maximus Planudes, lived apparently under the Emperor Andronicus II (1282-1328) and perhaps under his predecessor Michael VIII (1261-82) also. A man of wide learning, he wrote (at the

instance of Nicolas Rhabdas, presently to be mentioned) a treatise on *magic squares*; he showed, that is, how the numbers 1, 2, 3 ... n^2 could be placed in the n^2 compartments of a square, divided like a chess-board into n^2 small squares, in such a way that the sum of the numbers in each horizontal and each vertical row of compartments, as well as in the rows forming the diagonals, is always the same, namely $\frac{1}{2}n(n^2 + 1)$. Moschopoulos gives rules of procedure for the cases in which $n = 2m + 1$ and $n = 4m$ respectively, and these only, in the treatise as we have it; he promises to give the case where $n = 4m + 2$ also, but does not seem to have done so, as the two manuscripts used by Tannery have after the first two cases the words *τέλος τοῦ αὐτοῦ*. The treatise was translated by De la Hire,¹ edited by S. Günther,² and finally edited in an improved text with translation by Tannery.³

The work of Moschopoulos was dedicated to Nicolas Artavasdus, called RHABDAS, a person of some importance in the history of Greek arithmetic. He edited, with some additions of his own, the Manual of Planudes; this edition exists in the Paris MS. 2428. But he is also the author of two letters which have been edited by Tannery in the Greek text with French translation.⁴ The date of Rhabdas is roughly fixed by means of a calculation of the date of Easter 'in the current year' contained in one of the letters, which shows that its date was 1341. It is remarkable that each of the two letters has a preface which (except for the words *τὴν δὴλωσιν τῶν ἐν τοῖς ἀριθμοῖς ζητημάτων* and the name or title of the person to whom it is addressed) copies word for word the first thirteen lines of the preface to Diophantus's *Arithmetica*, a piece of plagiarism which, if it does not say much for the literary resource of Rhabdas, may indicate that he had studied Diophantus. The first of the two letters has the heading 'A concise and most clear exposition of the science of calculation written at Byzantium of Constantino, by Nicolas Artavasdus

¹ *Mém. de l'Acad. Royale des Sciences*, 1705.

² *Vermischte Untersuchungen zur Gesch. d. Math.*, Leipzig, 1876.

³ 'Le traité de Manuel Moschopoulos sur les carrés magiques' in *Annuaire de l'Association pour l'encouragement des études grecques*, **xx**, 1886, pp. 88-118.

⁴ 'Notices sur les deux lettres arithmétiques de Nicolas Rhabdas' in *Notices et extraits des manuscrits de la Bibliothèque Nationale*, **xxii**, pt. 1, 1886, pp. 121-252.

Smyrna, arithmetician and geometer, τοῦ 'Ραβδᾶ, at the instance of the most revered Master of Requests, Georgius utzyces, and most easy for those who desire to study it.' This long passage, called ἑκφρασις τοῦ δακτυλικοῦ μέτρου, deals with a method of finger-notation, in which the fingers of each hand held in different positions are made to represent numbers.¹ The fingers of the left hand serve to represent all the units and tens, those of the right all the hundreds and thousands up to 9000; 'for numbers above these it is necessary to use writing, the hands not sufficing to represent such numbers.' The numbers begin with the little fingers of each hand; if we call the thumb and the fingers after it the 1st, 2nd, 3rd, 4th, and 5th fingers in the German style, the successive signs may be thus described, premising that, where fingers are not either bent or 'half-closed' (κλινόμενοι) or 'closed' (συστελλόμενοι), they are supposed to be held out straight (τεινόμενοι).

(a) *On the left hand:*

for 1, half-close the 5th finger only;

" 2, " " 4th and 5th fingers only;

" 3, " " 3rd, 4th and 5th fingers only;

" 4, " " 3rd and 4th fingers only;

" 5, " " 3rd finger only;

" 6, " " 4th " "

" 7, close the 5th finger only;

" 8, " " 4th and 5th fingers only;

" 9, " " 3rd, 4th and 5th fingers only.

(b) The same operations on the *right hand* give the thousands, from 1000 to 9000.

(c) *On the left hand:*

for 10, apply the tip of the forefinger to the first joint of the thumb so that the resulting figure resembles σ;

¹ A similar description occurs in the works of the Venerable Bede (*De computo vel loquela digitorum*), forming chapter i of *De temporum ratione*, where expressions are also quoted from St. Jerome (d. 420 A.D.) showing that he too was acquainted with the system (*The Miscellaneous Works of the Venerable Bede*, ed. J. A. Giles, vol. vi, 1848, pp. 141-8).

for 20, stretch out the forefinger straight and vertical, keep fingers 3, 4, 5 together but separate from it and inclined slightly to the palm; in this position touch the forefinger with the thumb;

„ 30, join the tips of the forefinger and thumb;

„ 40, place the thumb on the knuckle of the forefinger behind, making a figure like the letter Γ ;

„ 50, make a like figure with the thumb on the knuckle of the forefinger *inside*;

„ 60, place the thumb inside the forefinger as for 50 and bring the forefinger down over the thumb, touching the ball of it;

„ 70, rest the forefinger round the tip of the thumb, making a curve like a spiral;

„ 80, fingers 3, 4, 5 being held together and inclined at an angle to the palm, put the thumb across the palm to touch the third phalanx of the middle finger (3) and in this position bend the forefinger above the first joint of the thumb;

„ 90, close the forefinger only as completely as possible.

(d) The same operations on the right hand give the hundreds, from 100 to 900.

The first letter also contains tables for addition and subtraction and for multiplication and division; as these are said to be the 'invention of Palamedes', we must suppose that such tables were in use from a remote antiquity. Lastly, the first letter contains a statement which, though applied to particular numbers, expresses a theorem to the effect that

$$(a_0 + 10a_1 + \dots + 10^m a_m) (b_0 + 10b_1 + \dots + 10^n b_n) \\ \text{is not } > 10^{m+n+2},$$

where $a_0, a_1 \dots b_0, b_1 \dots$ are any numbers from 0 to 9.

In the second letter of Rhabdas we find simple algebraical problems of the same sort as those of the *Anthologia Graeca* and the Papyrus of Akhmim. Thus there are five problems leading to equations of the type

$$\frac{x}{m} + \frac{x}{n} + \dots = a.$$

Rhabdas solves the equation $\frac{x}{m} + \frac{x}{n} = a$, practically as we could, by multiplying up to get rid of fractions, whence he obtains $x = mna/(m+n)$. Again he solves the simultaneous equations $x+y=a$, $mx=ny$; also the pair of equations

$$x + \frac{y}{m} = y + \frac{x}{n} = a.$$

course, m , n , a ... have particular numerical values in cases.

Rhabdas's Rule for approximating to the square root of a non-square number.

We find in Rhabdas the equivalent of the Heronian formula for the approximation to the square root of a non-square number $A = a^2 + b$, namely

$$\alpha = a + \frac{b}{2a};$$

he further observes that, if α be an approximation by excess, then $\alpha_1 = A/\alpha$ is an approximation by defect, and $\frac{1}{2}(\alpha + \alpha_1)$ is an approximation nearer than either. This last form is of course exactly Heron's formula $\alpha = \frac{1}{2}\left(a + \frac{A}{a}\right)$. The formula is also known to Barlaam (presently to be mentioned), who also indicates that the procedure can be continued indefinitely. It should here be added that there is interesting evidence of the Greek methods of approximating to square roots in two documents published by Heiberg in 1899.¹ The first of these documents (from a manuscript of the fifteenth century from Vienna) gives the approximate square root of certain non-square numbers from 2 to 147 in integers and proper fractions. The numerals are the Greek alphabetic numerals, but they are even place-value like our numerals: thus $\alpha\eta = 18$, $\alpha\delta\zeta = 147$, $\frac{13}{28}$, and so on: 0 is indicated by η or, sometimes, by \circ . Of these square roots, such as $\sqrt{(21)} = 4\frac{2}{3}\frac{1}{8}$, $\sqrt{(35)} = 5\frac{1}{2}\frac{1}{2}$, $\sqrt{(112)} = 10\frac{3}{4}\frac{3}{4}$, and so on, can be obtained (either exactly or, in a few cases, by neglecting or adding a small fraction in the process).
¹ 'Byzantinische Analekten' in *Abh. zur Gesch. d. Math.* ix. Heft, 1899, pp. 168 sqq.

numerator of the fractional part of the root) in one or other of the following ways:

(1) by taking the nearest square to the given number A , say a^2 , and using the Heronian formulae

$$\alpha_1 = \frac{1}{2} \left(a + \frac{A}{a} \right), \quad \alpha_2 = \frac{1}{2} \left(\alpha_1 + \frac{A}{\alpha_1} \right), \text{ \&c.};$$

(2) by using one or other of the following approximations, where

$$a^2 < A < (a+1)^2, \text{ and } A = a^2 + b = (a+1)^2 - c,$$

namely,

$$a + \frac{b}{2a}, \quad a + \frac{b}{2a} + \frac{b}{2a},$$

$$(a+1) - \frac{c}{2(a+1)}, \quad (a+1) - \frac{c}{2(a+1)} - \frac{c}{2(a+1)},$$

or a combination of two of these with

(3) the formula that, if $\frac{a}{b} < \frac{c}{d}$, then

$$\frac{a}{b} < \frac{ma+nc}{mb+nd} < \frac{c}{d}.$$

It is clear that it is impossible to deny to the Greeks the knowledge of these simple formulae.

Three more names and we have done.

IOANNES PEDIASIMUS, also called Galenus, was Keeper of the Seal to the Patriarch of Constantinople in the reign of Andronicus III (1328-41). Besides literary works of his, some notes on difficult points in arithmetic and a treatise on the duplication of the cube by him are said to exist in manuscripts. His *Geometry*, which was edited by Friedlein in 1866, follows very closely the mensuration of Heron.

BARLAAM, a monk of Calabria, was abbot at Constantinople and later Bishop of Geraci in the neighbourhood of Naples; he died in 1348. He wrote, in Greek, arithmetical demonstrations of propositions in Euclid, Book II,¹ and a *Logistic* in six Books, a laborious manual of calculation in whole numbers,

¹ Edited with Latin translation by Dasypodius in 1564, and included in Heiberg and Menge's Euclid, vol. v, *ad fin.*

inary fractions and sexagesimal fractions (printed at Crassburg in 1592 and at Paris in 1600). Barlaam, as we have seen, knew the Heronian formulae for finding successive approximations to square roots, and was aware that they could be indefinitely continued.

ISAAC ARGYRUS, a monk, who lived before 1368, was one of a number of Byzantine translators of Persian astronomical works. In mathematics he wrote a *Geodiaesia* and scholia to the first six Books of Euclid's *Elements*. The former is contained in the Paris MS. 2428 and is called 'a method of codesy or the measurement of surfaces, exact and shortened'; the introductory letter addressed to one Colybos is followed by a compilation of extracts from the *Geometrica* and *Stereometrica* of Heron. He is apparently the author of some further additions to Rhobdas's revision of the Manual of planudes contained in the same manuscript. A short tract of his 'On the discovery of the square roots of non-rational square numbers' is mentioned as contained in two other manuscripts at Venice and Rome respectively (Codd. Marcianus Gr. 33 and Vaticanus Gr. 1058), where it is followed by a table of the square roots of all numbers from 1 to 102 in sexagesimal fractions (e.g. $\sqrt{2} = 1\ 24' 51'' 48'''$, $\sqrt{3} = 1\ 43' 56'' 0'''$).¹

¹ Heiberg, 'Byzantinische Annalekten', in *Abh. zur Gesch. d. Math.* ix, p. 169-70.

APPENDIX

On Archimedes's proof of the subtangent-property of a spiral.

THE section of the treatise *On Spirals* from Prop. 3 to Prop. 20 is an elaborate series of propositions leading up to the proof of the fundamental property of the subtangent corresponding to the tangent at any point on any turn of the spiral. Libri, doubtless with this series of propositions in mind, remarks (*Histoire des sciences mathématiques en Italie*, i, p. 31) that 'Après vingt siècles de travaux et de découvertes, les intelligences les plus puissantes viennent encore échouer contre la synthèse difficile du *Traité des Spirales* d'Archimède.' There is no foundation for this statement, which seems to be a too hasty generalization from a dictum, apparently of Fontenelle, in the *Histoire de l'Académie des Sciences pour l'année 1704* (p. 42 of the edition of 1722), who says of the proofs of Archimedes in the work *On Spirals*: 'Elles sont si longues, et si difficiles à embrasser, que, comme on l'a pu voir dans la Préface de l'Analyse des Infiniment petits, M. Bouillaud a avoué qu'il ne les avoit jamais bien entendues, et que Viète les a injustement soupçonnées de paralogisme, parce qu'il n'avoit pu non plus parvenir à les bien entendre. Mais toutes les preuves qu'on peut donner de leur difficulté et de leur obscurité tournent à la gloire d'Archimède; car quelle vigueur d'esprit, quelle quantité de vûes différentes, quelle opiniâtreté de travail n'a-t-il pas fallu pour lier et pour disposer un raisonnement que quelques-uns de nos plus grands géomètres ne peuvent suivre, tout lié et tout disposé qu'il est?'

P. Tannery has observed¹ that, as a matter of fact, no mathematicians of real authority who have applied or extended Archimedes's methods (such men as Huygens, Pascal, Roberval and Fermat, who alone could have expressed an opinion worth having), have ever complained of the

¹ *Bulletin des sciences mathématiques*, 1895, Part i, pp. 265-71.

city' of Archimedes; while, as regards Vieta, he has that the statement quoted is based on an entire mis-
 en- sion, and that, so far from suspecting a fallacy in
 nes's proofs, Vieta made a special study of the treatise
irals and had the greatest admiration for that work.

as in many cases in Greek geometry where the analy-
 omitted or even (as Wallis was tempted to suppose) of
 pose hidden, the reading of the completed synthetical
 leaves a certain impression of mystery; for there is
 g in it to show *why* Archimedes should have taken
 ly this line of argument, or how he evolved it. It is
 that, as Pappus said, the subtangent-property can be
 shed by purely 'plane' methods, without recourse to
 d' *velours* (whether actually solved or merely assumed
 e of being solved). If, then, Archimedes chose the more
 t method which we actually find him employing, it is
 y possible to assign any reason except his definite
 ection for the form of proof by *reductio ad absurdum*.
 ultimately on his famous 'Lemma' or Axiom.

seems worth while to re-examine the whole question of
 sccovery and proof of the property, and to see how
 nes's argument compares with an easier 'plane' proof
 ted by the figures of some of the very propositions
 d by Archimedes in the treatise.

he first place, we may be sure that the property was
 scovered by the steps leading to the proof as it stands.
 ot but think that Archimedes divined the result by an
 ent corresponding to our use of the differential calculus
 etermining tangents. He must have considered the
 taneous direction of the motion of the point P describ-
 e spiral, using for this purpose the parallelogram of
 lies. The motion of P is compounded of two motions,
 long OP and the other at right angles to it. Comparing
 stances traversed in an instant of time in the two direc-
 we see that, corresponding to a small increase in the
 vector r , we have a small distance traversed perpen-
 rly to it, a tiny arc of a circle of radius r subtended by
 gle representing the simultaneous small increase of the
 θ ($\angle AOP$). Now r has a constant ratio to θ which we call
 en θ is the circular measure of the angle θ). Consequently

(arc ASP) = OT , by hypothesis ;

so it was necessary to prove, *alternando*, that

(arc RP) < $RO : OT$, or $PO : OT$,

i.e. < $PM : MO$, where OM is perpendicular to SP .

And, in order to satisfy (2), it was necessary to

show that

$$PQ : (\text{arc } PQ) > PM : MO.$$

As a matter of fact, (3) is *a fortiori* satisfied if

$P'R : (\text{chord } RP) < PM : MO$;

In the case of (4) we cannot substitute the chord PQ for the arc PQ , and we have to substitute PG' , where G' is the point in which the tangent at P to the circle meets OQ produced ; for of course $PG' > (\text{arc } PQ)$, so that (4) is *a fortiori* satisfied if

$$PQ : PG' > PM : MO.$$

It is remarkable that Archimedes uses for his proof of the two cases Prop. 8 and Prop. 7 respectively, and makes no use of Props 6 and 9, whereas the argument points precisely to the use of the figures of the latter propositions only.

In the figure of Prop. 6 (Fig. 1), if OPP' is any radius drawn from O to P , and if PB produced cuts OT , the parallel to OP through O , in H , it is obvious, by parallels, that

$$PP' : (\text{chord } PB) = OP : PH.$$

As PH becomes greater the farther P moves from B towards O , so that the ratio $PP' : PB$ diminishes continually, it is always less than $OB : BT'$ (where BT' is the tangent from B to OH in T'), i.e. always less than $BM : MO$. The relation (9) is always satisfied for any point R' of the circle on the 'backward' side of P .

This is equivalent to (1), from which it follows that $P'R$ is always less than RR' , so that R' always lies on the side of O .

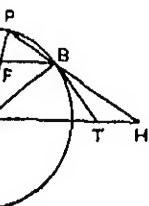


FIG. 1.

aches that ratio without limit as R approaches P . But the proof does not enable us to say that $RF' : (\text{chord } PR)$, which is $> RF' : PG$, is also always less than $PM : MO$. At first sight, therefore, it would seem that the proof must fail. So, however; Archimedes is nevertheless able to prove that if PV and not PT is the tangent at P to the spiral, an equality follows. For his proof establishes that, if PV is the tangent and OF'' is drawn as in the proposition, then

$$F'O : RO < OR' : OP,$$

but $OR' : OP < OR' : OR'$, 'which is impossible'. Why this is impossible does not appear in Props. 18 and 20, but it follows from the argument in Prop. 13, which proves that a tangent to the spiral does not meet the curve again, and in fact that the spiral is everywhere concave towards the origin.

Similar remarks apply to the proof by Archimedes of the impossibility of the other alternative supposition (that the tangent at P meets OT' at a point U nearer to O than T is).

Archimedes's proof is therefore in both parts perfectly valid, free of any appearances to the contrary. The only drawback that can be urged seems to be that, if we assume the tangent to cut OT' at a point *very near* to T on either side, Archimedes's construction brings us perilously near to infinitesimals, and the proof may appear to hang, as it were, on a thread, albeit a thread strong enough to carry it. But it is remarkable that he should have elaborated such a difficult proof by means of Props. 7, 8 (including the 'solid' *νεῦσις* of Prop. 8), when the figures of Props. 6 and 7 (or 9) themselves establish the direct proof above given, which is independent of the *νεῦσις*.

Tannery,¹ in a paper on Pappus's criticism of the proof as necessarily involving 'solid' methods, has given another proof of the subtangent-property based on 'plane' methods, but I prefer the method which I have given above, since it corresponds more closely to the preliminary proposition actually given by Archimedes.

¹ Tannery, *Mémoires scientifiques*, i, 1912, pp. 300-16.

INDEX OF GREEK WORDS

pages are those of the first volume except where otherwise stated.]

- ἄκτιον 47.
 ῥητος, -ον: ἀγεωμέτρητος μη-
 λίστω (Plato) iii. 355.
 -τος, -ον, that cannot be gone
 through, i.e. infinite 343.
 -ον ii. 462: ἀπαγωγή εἰς
 τὸν, &c. 372.
 postulato 873.
 ἄκτιος 11.
 -ον, irrational 84, 90: περὶ
 τῶν γραμμῶν καὶ κασπῶν (Demo-
 critus) 156-7, 181: ἄλλοι οὐκ ἔστιν
 αὐτὸν (Plato) 157.
 αὐτὸν ii. 287.
 -ον, proportional: used as ad-
 jective 85.
 ἄνθος (τόπος), 'Treasury of
 the gods' 421-2, ii. 399, 400, ii.
 401.
 -ον, inversely 385: ἀντίπαλον
 ii. 400.
 ἄνθος (ἀναστρέφω), conver-
 sion 386.
 ἄνθος, conversion ib.
 ἄνθος (τόπος), a class of locus
 385.
 ἄνθος by Hypsicles 419, ii. 213.
 αὐτὸν ii. 244.
 αὐτὸν 341.
 -ον, undefined: πλῆθος μα-
 θημάτων (= unknown, x)
 456: ἐν ἀντίπαλῳ ii. 480, 491.
 ἄνθος, reduction 872: ἀπ. εἰς
 τὸν, *reductio ad absurdum*
 873.
 ἄνθος, proof 870, ii. 538.
 ἄνθος, distance or dimension
 873, or interval 806 n.
 ἄνθος, 'shoemaker's knife' ii. 23,
 31-2, ii. 371-7.
 ἄνθος, theory of numbers, opp.
 ἄνθος 13-16.
 ἀριθμητικός, -ή, -όν: ἀριθμητικὴ εἰσα-
 γωγή of Nicomachus 97.
 ἀριθμός, number: definitions of
 'number' 60-70: in Diophantus,
 used for unknown quantity (x)
 94, ii. 453.
 ἀριθμοστόν: reciprocal of ἀριθμός
 ($=x$) in Diophantus ii. 458.
 ἀρπεδονίπται, 'rope-stretchers' 121-
 2, 178.
 ἀρρητος, -ον, irrational 157.
 ἀρτιάκις ἄρτιος, even-times-even 71,
 with Neo-Pythagoreans = 2", 72.
 ἀρτιάκις περιττός, even-times-odd 72.
 ἀρτιοπέριτος, even-odd, restricted
 by Neo-Pythagoreans to form
 $2(2m+1)$, 72.
 ἄρτιος, -α, -ον, even 70.
 Ἀρχαί, a lost work of Archimedes
 ii. 81.
 Ἀστροθεσίαι of Eratosthenes ii. 109.
 ἀστρολάβον ὄργανον of Hipparchus
 ii. 256.
 ἀσύμμετρος, -ον, incommensurable
 157.
 ἀσύμπτωτος, -ον, non-secant ii. 227.
 ἀσύνθετος, -ον, incompressible 72.
 ἄτομος, -ον, indivisible 181: Aristo-
 telian περὶ ἀτόμων γραμμῶν 157,
 346-8.
 ἄτοπος, -ον, absurd ii. 462.
 αὐξάνειν: τρις αὐξήσεις (Plato) 306-7.
 αὐξή, τρις, 297: κύβων αὐξή, 297.
 αὐξήσις 805-8 n.
 αὐτοματοποιητική ii. 808.
 ἀψίς, segment of circle less than a
 semicircle ii. 314.
 Βαρουλκός of Heron ii. 809, ii. 346-7.
 Βελοπαυκός of Heron 18, ii. 298, ii.
 302, ii. 308-9.
 βιάζειν: βίβιασμένος, forced or un-
 natural ii. 362.

βωμίσκος, 'little altar', properly a wedge-shaped solid ii. 319, ii. 333: measurement of (Heron), ii. 332-3: (= σφηνίσκος) of a certain kind of solid number 107, ii. 240, ii. 315.

γεωδυσία = mensuration 16.

Γεωμετρούμενα of Heron ii. 318, ii. 453.

γλῶχis (arrow-head), Pythagorean name for angle 166.

γνωμονική 18.

γνώμων, *gnomon*, q.v.: κατὰ γνώμονα = perpendicular 78, 175.

γνώριμος, -ον, known: γνώριμον, an alternative term for δεδομένον, given ii. 537.

γνωρίως, 'in the recognized manner' ii. 79.

γράμμα, 'figure' or proposition, of theorem of Eucl. I. 47, 144.

γραμμή: διὰ or ἐκ τῶν γραμμῶν of theoretical proof ii. 257, 258.

γραμμικός, -ή, -όν, linear: used of prime numbers 73: γραμμικαὶ ἐπιστάσεις, 'Considerations on Curves', by Demetrius ii. 359: γραμμικῶς, graphically 93.

γράφειν, to draw or write on 159, 178: also to prove 203 n., 339.

δεδομένος, -η, -ον, given: senses of, ii. 537-8.

δεικνύναι, to prove 328.

δεῖν: δεῖ δὴ 371.

δεύτερος, *secondary*: of composite numbers 72: δευτέρα μυριάς (= 10,000²) 40.

διαβήτης, compasses 308, ii. 540.

διαίρειν: διελόντι, *separando* or *dividendo* (in transformation of ratios) 386.

διαίρεσις: λόγου, *separation* of a ratio 386: περὶ διαίρεσεων βιβλίον, *On divisions (of figures)*, by Euclid 425.

διαστάσις, dimension: περὶ διαστάσεως, a work of Ptolemy ii. 295.

διάστημα, interval 215: distance 239.

διὰυλος, 'race-course': representations of square and oblong numbers as sums of terms 114.

διδόναι: δεδομένον, *given*, senses ii. 537-8.

διεξοδικός (τόπος), a species of locus ii. 185.

διῆσταναι: ἐφ' ἐν διεστῶς, extended one way ii. 428.

δικόλουρος, -ον, twice-truncated 107.

διοπτρα, dioptra, q.v.

διοπτρική 18.

διορίζειν: διορισμένη τομή, *Determinate Section*, by Apollonius ii. 180.

διορισμός, definition, delimitation: two senses (1) a constituent part of a theorem or problem 370, (2) a statement of conditions of possibility of a problem 303, 319-20, 371, 377, 395, 398, 428, ii. 45-6, ii. 129-32, ii. 168, ii. 230.

διπλοῖσότης, double-equation (Diophantus) ii. 468.

διπλοῦς, -ῆ, -οῦν: διπλῆ μυριάς = 10,000² (Apollonius) 40: διπλῆ ἰσότης, διπλῆ ἰσῶσις, double-equation (Diophantus) ii. 468.

δοκίς, *beam*, a class of solid number 107, ii. 240.

δοκός = δοκίς ii. 315.

δραχμή, sign for, 31, 49, 50.

δύναμις: incommensurable side of square containing a non-square number of units of area 208-4: square or square root 209 n., 297: square of unknown quantity (= x^2) (Diophantus) ii. 457-8: δυναμίς, 'in square' 187, 308: τετραπλῆ δύναμις = eighth power (Egypt) ii. 546: power in mechanics 445.

δυναμοδύναμις, square-square = fourth power (Heron) ii. 458: fourth power of unknown (Diophantus) ii. 458, ii. 546.

δυναμόκυβος, square-cube, = fifth power of unknown (Diophantus) ii. 458.

δυναμοστόν, δυναμοδυναμοστόν, &c., reciprocals of powers of unknown (Diophantus) ii. 458.

δύνασθαι, to be equivalent 'in square' to, i.e. to be the side of a square equal to (a given area): δυναμένη 305-6 n.

δυναστευομένη, opp. to δυναμένη 305-6 n.

εἶδος, 'figure' of a conic ii. 139: 'species' = particular power

of unknown, or term, in an equation (Diophantus) ii. 460.
 μία, *én*, one: *ἑνὶ πλείω*, 'several ones' (definition of 'number') 70.
ἡγεῖσθαι, to introduce or explain 213.
ἑστίς, setting-out 370, ii. 533.
ἑπερίσματα of Democritus 178, 181.
ἑμμετρος (κύκλος) ii. 288.
ἑκψις, falling-short (in application of *ἀρεταί*), name given to *ellipse* by Apollonius 150, ii. 138.
ἑκπύς, -*ης*, defective (of numbers), contrasted with *perfect* 74, 101: *ἑκπύς καὶ τὸν νεῦον* ii. 459.
ἑλπίς, alternately (in proportions) 385.
ἑκπύς, notion: *κοινὰ ἔννοια*, common notions = *axioms* 386.
ἑκπύς, objection 372, ii. 311, ii. 533.
ἑκπύς, bulging out 6.
ἑκπύς ii. 234.
ἑκπύς, elucidation ii. 223, ii. 231-2.
ἑκπύς, or *πρώτον ἑξ.*, a 60th (= minute), *δεύτερον ἑξ.*, a second, &c. 45.
ἑκπύς, ('bloom') of Thymaridas: a system of linear equations solved 14.
ἑκπύς, contact: *ἑκπύς*, *Contacts or Tangencies*, by Apollonius ii. 81.
ἑκπύς, on: *τὸ σημῆιον ἐφ' ᾧ* (or *ἐν*) *K*, archaic for 'the point *K*' 199: *ἐφ' ᾧ AB*, 'the straight line *AB*' *ib.*
ἑκπύς, *superpartiens*,
 = ratio $1 + \frac{1}{m+n}$, 102.
ἑκπύς, *superparticularis* = ratio of form $(n+1)/n$, 90, 101: *ἑκπύς* *ἀριον* *διόστημα* 215.
ἑκπύς, *μετεωρικά* ii. 453.
ἑκπύς, weather indications 77 n., ii. 284.
ἑκπύς = ratio $4/3$, 101: *ἑκπύς* *ὑπομήνη* (Plato) 306-7.
ἑκπύς, *τὰ ἑσχατα*, extremities 93.
ἑκπύς, -*ης*, *oblong*: of numbers of form $m(m+1)$, 82, 108.
ἑκπύς (ἀριθμός) = *prime* 72.

ἑκπύς, a class of locus ii. 185, ii. 193.
ἑκπύς, *Method* ii. 246.
ἑκπύς, lever or balance: *περὶ ζυγῶν*, a work of Archimedes ii. 23-4, ii. 351.
ἑκπύς, -*ης*, -*ον*, ratio of $3/2$, 101.
ἑκπύς, *ἡμιωβελίον*, $\frac{1}{2}$ -obol, sign for 81, 49, 50.
ἑκπύς 18.
ἑκπύς, *θεολογοῦμενα ἀριθμητικῆς* 97.
ἑκπύς, position: *παρὰ θέσει* (sc. *δεδομένην*), parallel to a straight line given in position ii. 193: *πρὸς θέσει*, *εὐθείας*, on straight lines given in position ii. 426.
ἑκπύς, *shield*, old name for *ellipse* 439, ii. 111, ii. 125.
ἑκπύς: *ἑκπύς* used by Plato of the earth 314-15.
ἑκπύς, equal an equal number of times, or equal multiplied by equal 204.
ἑκπύς, -*ον*, of equal contour: *περὶ ἑκπύων σφαιρῶν*, by Zeno-dorus ii. 207, ii. 390.
ἑκπύς, -*ον*, equilateral: of square number (Plato) 204.
ἑκπύς, equilibrium: *περὶ ἑκπύων*, work by Archimedes ii. 24, ii. 351.
ἑκπύς, equal: *δι' ἑκπύων*, *ex aequali* (in proportions) 386: *δι' ἑκπύων ἐν τετραγώνῳ* 386.
ἑκπύς, or *ἑκπύς*, equation ii. 468.
ἑκπύς, inquiry, Pythagoras's name for geometry 186.
ἑκπύς, power (in mechanics) 445.
ἑκπύς, turning-point in race-course 114.
ἑκπύς, -*ης*, -*ον*, curved 249, 341.
ἑκπύς, *Canon*, q.v.
ἑκπύς, ruler 239: *Table* (astron.), *Προχέλρων κανόνων διάταξις καὶ ψηφισμοί*, work by Ptolemy ii. 293: *canon* (in music), v. *Καταμετρή.*
ἑκπύς, to inscribe-in or on (c. gen.) 181.
ἑκπύς, work by Eratosthenes ii. 108.
ἑκπύς, 193 n.
ἑκπύς, construction (constituent part of proposition) 370, ii. 533.

Κατωστερισμοί, work by Eutosthenes ii. 108.

Κατατομή κανόνος, *Sectio canonis*, attributed to Euclid 17, 444.

κατονώμαξις τῶν ἀριθμῶν, naming of numbers (Archimedes) ii. 23.

κατοπτρική, theory of mirrors 18.

κεντροβαρικά, problems on centre of gravity ii. 24, ii. 350.

κέντρον, centre: ἢ ἐκ τοῦ κέντρου = radius 381.

κερατοειδής (γωνίῳ) 178, 382.

Κηρίν of Sporus 234.

κλίειν, infect: κεκλῆσθαι 337.

κογχοειδής γραμμή, *conchoid* 238.

κοιλογώνιον ii. 211.

κόλυρος, -ον, truncated ii. 333: (of pyramidal number) 107.

κόσκιον, *sieve* (of Eratosthenes) 16, 100, ii. 105.

κοχλοειδής γραμμή, *cochloid* 238.

κυβήκυβος, cube-cube, = sixth power of unknown (Diophantus) ii. 458.

κυβοκυβοστόν, reciprocal of κυβήκυβος ii. 458.

κύβιν, cube: κύβων ἀδξή (Plato) 297: cube of unknown (Diophantus) ii. 458: κύβος ἐξελικτός = ninth power of unknown (Egyptian) ii. 546.

κυκλική θεωρία, *De motu circulari*, by Cleomedes ii. 235.

κυκλικός, -ή, -όν, circular, used of square numbers ending in 5 or 6, 108.

λείπειν: forms used to express *minus*, and sign for (Diophantus), ii. 459.

λείψις, *wanting* (Diophantus): λείψει = *minus* ii. 459.

λέξις: κατὰ λέξιν, word for word 183.

λεπτόν, a *fraction* (Heron) 43: = a *minute* (Ptolemy) 45.

λήμμα, *lemma* 378.

λογισμός, calculation 13.

λογιστική, art of calculation, opp. to ἀριθμητική 13-16, 53.

λόγος, ratio: λόγων ἀποτομή, *sectio rationis*, by Apollonius ii. 173.

μαθήματα, subjects of instruction 10-11: term first appropriated to mathematics by Pythagoreans 11: περί τῶν μαθημάτων, a work by Protagoras 179.

μαθηματικός, -ή, -όν: μαθηματικοί in Pythagorean school, opp. to ἀκουσματικοί 11: Μαθηματική σύνταξις of Ptolemy ii. 273-4: μαθηματικά, τὰ (Plato) 288.

μεθόριον, boundary ii. 449.

μελουργον προεσκαρπιφευμένον (Heron), curtailed and paved in front (cf. scarify), of a long, narrow, triangular prism (Heib.) ii. 319.

μέρος: μέρος, parts (= proper fraction) dist. from μέρος (aliquot part) 42 (cf. p. 294).

μεσολύβον, *mean-finder* (of Eratosthenes) ii. 104, ii. 359.

μετέωρος, -ον: περί μετεώρων, work by Posidonius ii. 219, ii. 231-2.

μετεωροσκοπική 18.

Μετρήσεις, *Mensurae* (Heronian) ii. 319.

μήκος, length: used by Plato of side of square containing a square number of units of area 204.

μηλίτης (ἀριθμός), term for problems about numbers of apples (e.g.) 14, ii. 442.

Μικρός ἄστρονομούμενος (τόπος), *Little Astronomy* ii. 273.

μνά, mina (= 1000 drachmae): ἄν stands for, 81.

μοῖρα, fraction: 1/360th of circumference or a *degree* 45, 61: μοῖρα τοπική, χρονική (in Hypsicles) ii. 214.

μονάς, monad or unit 43: definitions of, 69: μονάδων σύστημα = number, 69: δευτερωδομένη μονάς = 10, τριωδομένη μ. = 100, &c. (Iamb.) 114: μονάς θέσιν ἔχουσα = point 69, 289.

μόριον, part or fraction: μορίου or ἐν μορίῳ = divided by (Diophantus) 44.

μυριάς (with or without πρώτη or ἀπλή) myriad (10,000), μ. δευτέρα or διπλή 10,000², &c. 40.

ναστόν (solid?) 156, 178.

νεύειν, to *verge* (towards) 196, 239, 337, ii. 65.

νεύσις, *inclination* or 'verging', a type of problem 235-41, 260, ii. 199, ii. 385: νεύσεις in Archimedes ii. 65-8: two books of νεύσεις by Apollonius ii. 189-92 ii. 401, ii. 412-13.

goal or end of race-course
 , obol : sign for, 31, 49, 50.
 ιονίκαι, work by Eratosthenes
 99.
 wedge-shaped figure ii. 319,
 33.
 ποιική 18.
 -α, -ον, right or perpendi-
 r: ὀρθία πλευρά, *latus rectum*
 39: ὀρθία διάμετρος, 'erect
 meter', in double hyperbola,
 84.
 : ὀρισμένος, defined, i. e. de-
 terminate 94, 340.
 (κύκλος), dividing circle :
 ζων (Eucl.) 351, 352.
 (1) definition 373: (2) limit
 boundary 293: (3) term (in a
 portion) 306 n.
 or οὐδέν, sign for (O), 39, 45.
 καὶ κινῶ τὰν γῆν, saying of
 himodes ii. 18.
 δύναται (αὐτὸν καταγόμεναι τετρα-
 γωνοῦ), expression for *parameter*
 ordinates ii. 139.
 ἡ, application: π. τῶν χωρῶν,
 lication of areas 150: τὰ ἐκ
 παραβολῆς γινόμενα σημεία, the
 of a central conic, ii. 156:
 ἀβολα (the conic) 150, ii. 138.
 ὀξυγράφησι ii. 541.
 ὀξὺς γραμμῇ, paradoxical curve
 Menelaus) ii. 260-1, ii. 360.
 ἡγῆμα 177, ii. 234.
 πᾶν, to pull awry: παρὰ πᾶ-
 νος ii. 398.
 ἴσης, nearness to equality, ap-
 proximation: παρσιδότητος ἀγωγή
 (pphantus) ii. 477, ii. 500.
 r, axe-shaped figure ii. 315.
 ζεῖν, to 'five' (= count) 26.
 ἄλος 176, ii. 104.
 οὐσα ποσότης = unit, 69.
 r, limit or extremity 293:
 iting surface 160: πέρας συγ-
 γων, definition of figure ii. 221.
 ἄρτιος, odd-even: with Neo-
 chagoreans is of form
 $2n+1$ ($2m+1$), 72.
 ὀδός, ἡ, -όν, odd, q. v.
 19.
 s, -η, -ον, how great (of mag-
 nitude) 12.

πληκτικῆς, size 384.
 πλάγιος, -α, -ον, transverse: πλαγία
 διάμετρος or πλευρά ii. 139.
 πλισματικός, -όν, (easily) formable
 ii. 487.
 Πλουτωνικός, a work by Eratosthenes
 ii. 104.
 πλήθος, multitude: πλήθος ἑν = unit,
 69: πλήθος ὀρισμένον = number,
 70: πλήθος μονάδων ὀριστον, def.
 of unknown 'quantity' 94, ii.
 456.
 πλυνθίς, a brick, a solid number of
 a certain form 107, ii. 240, ii.
 315.
 πολλαπλασιεπιμερής, *multiplex super-*
partiens, = ratio of form

$$p + \frac{m}{m+n}, 103.$$

 πολλαπλασιεπιμέριος, *multiplex su-*
perparticularis, = ratio of form

$$m + \frac{1}{n}, 103.$$

 πολλαπλίσσιος, -α, -ον, multiple 101.
 πολύσπαστος, a compound pulley ii.
 18.
 πόριμος, -ον (πορῖζειν), procurable :
 one sense of δεδομένος ii. 533.
 πόρισμα, porism: (1) = corollary,
 (2) a certain type of proposition
 372-3, ii. 533.
 ποσόν, quantity, of number, 12.
 ποσότης, quantity 69, 70: number
 defined as ποσότητος χύμα ἐκ μονά-
 δων συγκείμενον 70.
 προμήκης, *prolate* (= oblong) 203:
 but distinguished from ἑτερομήκης
 83, 108.
 προσαγωγήον 309.
 πρότυσις = enunciation 370, ii.
 533.
 πρώτος, prime 72.
 πτώσις, case 372.
 πυθμήν, base; = digit 55-7, 115-17:
 ἑπτέριος πυθμήν 306-7.
 πυραμῖς, pyramid 126.
 πύριον, πύριον, burning mirror:
 περὶ πυρίων, work by Dioctes
 264, ii. 200; περὶ τοῦ πυρίου, by
 Apollonius ii. 194.
 ῥητός, ἡ, -όν, rational: used in sense
 of 'given' ii. 537.
 ῥοπή: περὶ ῥοπῶν, a mechanical
 work by Ptolemy ii. 295.

αἰλινον of Archimedes ii. 23, ii. 103.

σῆκωμα 49.

σκιφή, a form of sun-dial ii. 1, ii. 4.

σκηνογραφική, scene-painting 18, ii. 224.

Σοφία, nickname of Democritus 176.

σπίρα, *spire* or *tore* ii. 117: varieties of (διεχής, συνεχής, ἐμπεπλεγμένη or ἐπαλλὰττοντα), ii. 204.

στάθμη, plumb-line 78, 309.

στατήρ, sign for, 31.

στερεομετρία, solid geometry 12-13.

στερεομετρούμενα ii. 453.

στήλις, column, a class of solid number, 107.

στιγμή, point 69: στιγμή ἄθετος = unit, 69.

στοιχείωτης, -ός, the writer of Elements (στοιχείων), used of Euclid 357.

στρογγύλος, -ον, round or circular 293.

συμπέρασμα, conclusion (of proposition) 370, ii. 533.

σύνθεσις (λόγου), composition (of a ratio) 385.

σύνταξις, collection: Μεγάλη σύνταξις of Ptolemy 348, called Μαθηματικὴ σύνταξις ii. 273.

συντιθέναι: συντίθεται = componendo (in proportion) 385.

σύστασις, construction 151, 158.

σφαιρικός, -ή, -όν, spherical: used of cube numbers ending in 5 or 6, 107-8.

σφακίσκος, stake, a form of solid number, 107.

σφηνίσκος, wedge, a solid of a certain form, measurement of, ii. 332-3: a solid number, 107, ii. 315, ii. 319.

σχέσις, relation 384.

σχηματοποιεῖν, to form a figure ii. 226.

Τάλαντον, sign for (T), 31, 50.

ταράσσειν: (δι' ἴσου) ἐν τεταραγμένῃ ἀναλογίᾳ, in disturbed proportion 386.

τίσσειν: τεταγμένον, assigned = datum ii. 192, ii. 537: αἱ καταγόμεναι τεταγμένως (εὐθείαι), (straight lines) drawn ordinate-wise = ordinates ii. 139: τεταγμένως κοτ-ῆχθαι ii. 134.

τάχος, speed: περί ταχών, work by Eudoxus 329.

τέλειος, -ον, -ον, perfect: τέλειος ἀριθμός 74, 101.

τετρασημέριον, $\frac{1}{2}$ of obol, sign for, 31, 49, 50.

τετραγωνίζειν, to square: ἡ τετραγωνίζουσα (γραμμὴ), the quadratrix 225, ii. 359.

τετραγωνισμός, squaring 173.

τετρακτὺς 75, 99 n., 313, ii. 241.

τετραπλὴ δύναμις = 8th power of unknown (Egyptian term) ii. 546.

τμήμα, segment: used of lunes as well as segments of circles 184: segments or sectors 187-9: τμήματα = $\frac{1}{360}$ th parts of circumference and $\frac{1}{120}$ th parts of diameter of circle (Ptolemy) 45.

τομεύς, shoemaker's knife, term for sector of circle 381.

τομή, section: τὰ περὶ τὴν τομήν (Proclus) 324-5.

τόπος, locus: classifications of loci 218-19, ii. 185: τόποι πρὸς γραμμαῖς, τόποι πρὸς ἐπιφανείαις (-ῳ) 218-19, 439: τόποι πρὸς μεσότητος ii. 105: τόπος ἀναλύμενος, Treasury of Analysis, q. v.

τόρνος, circle-drawer 78, 308.

τρίγωνος ἀριθμός, triangular number, 15-16.

τρίκλινος, thrice-truncated 107.

τρίπλευρον, three-side, Menelaus's term for spherical triangle ii. 262.

τριώβολον, sign for, 49.

ὕδρια ὠροσκοπεῖα, water-clocks ii. 309.

ὑπαρξίς, forthcoming: positive term, dist. from negative (λείψις) ii. 459.

ὑπερμερής, subsuperpartiens, reciprocal of ἐπιμερής 102.

ὑπερμώριος, subsuperparticularis, reciprocal of ἐπιμώριος 101.

ὑπερβολή, exceeding (in application of areas): name given to hyperbola 150, ii. 138.

ὑπερίτελος, ὑπερτελής, over-perfect (number) 74, 100.

Ὑποθίσεις τῶν πλανωμένων, work by Ptolemy ii. 298.

ὑποπολλαπλασίσις, ὑποπολλαπλασιεπι-

ής, ὑποπαλλυπλιστικὸς ὁμοιος, &c.
1-3.

ίνειν, subtend 193 n.

ήξ, starting-point (of measure) 114.

ς ἀπλανῶν ἀστέρων, work by Ptolemy, ii. 293.

της (ἀριθμός), (number) of howls (simple algebraical problems) ii. 442.

ηλία, by Geminus ii. 223.

ος ($\frac{1}{2}$ th of obol), sign for, 31 : 50.

χείρ, *manus*, in sense of number of men 27.

χειροβάλλιστρα ii. 309.

χρoιά, colour or skin: Pythagorean name for surface 166, 293.

Χρονογραφία, work by Eratosthenes ii. 109.

χρῶμα, colour (in relation to surface) 293.

χωρίον, area 300 n.: *χωρίον ὑποτομῆς*, *sectio spatii*, by Apollonius ii. 179.

Ψηφοφορία κατ' ἰσοδύς ii. 546.

᾽Οκυτάκιον of Apollonius 234, ii. 194. ii. 253.

ENGLISH INDEX

[The pages are those of the first volume except where otherwise stated.]

- Abacus 46-8.
 'Abdelmelik al-Shirāzī ii. 128.
 Abraham Echellensis ii. 127.
 Abū Bekr Muh. b. al-Ḥasan al-Karkhī, *see* al-Karkhī.
 Abū 'l Fath al-Iṣṭabūnī ii. 127.
 Abū 'l Wafā al-Būzjānī ii. 328, ii. 450, ii. 458.
 Abū Naṣr Mansūr ii. 262.
 Achilles of Zenō 275-6, 278-80.
 Adam, James, 305-7, 313.
 Addition in Greek notation 52.
 Adrastus ii. 241, 243, 244.
 Aëtius 158-9, 163, ii. 2.
 'Aganis': attempt to prove parallel-postulate 358, ii. 228-30.
 Agatharchus 174.
 Ahmes (Papyrus Rhind) 125, 130, ii. 441.
 Akhmīm, Papyrus of, ii. 543-5.
 Albertus Pius ii. 26.
 Al-Chūzīnī ii. 260-1.
 Alexander the 'Actolian' ii. 242.
 Alexander Aphrodisiensis 184, 185, 186, 222, 223, ii. 223, ii. 231.
 Alexeieff, ii. 324-5 n.
 Al-Fakhri, by al-Karkhī 109, ii. 449-50.
 Algebra: beginnings in Egypt ii. 440: *hau*-calculations ii. 440-1: Pythagorean, 91-7: *epanthema* of Thymaridas 94-6.
 Algebra, geometrical, 150-4: application of areas (q.v.) 150-3: scope of geometrical algebra 153-4: method of proportion *ib.*
 Al-Ḥajjāj, translator of Euclid, 362: of Ptolemy ii. 274.
 Alhazen, problem of, ii. 294.
 Al-Kāfī of al-Karkhī 111.
 Al-Karkhī: on sum of
 $1^3 + 2^3 + \dots + n^3$
 109-10, 111, ii. 51, ii. 449.
 Allman, G. J. 184, 183.
Almagest ii. 271.
 Alphabet, Greek: derived from Phoenician, 31-2: Milesian, 33-4: quasi-numerical use of alphabet, 35-6 n.
 Alphabetic numerals 31-40, 42-4.
 Amasis 4, 129.
 Amenemhat I 122, III 122.
 Ameristus 140, 141, 171.
 Amyclas (better Amyntas) 320-1.
 Amyntas 320-1.
Analemma of Ptolemy ii. 280-92: of Diodorus ii. 287.
 Analysis: already used by Pythagoreans 168: supposed invention by Plato 291-2: absent from Euclid's *Elements* 371-2: defined by Pappus ii. 400.
 Anatolius 11, 14, 97, ii. 448, ii. 545-6.
 Anaxagoras: explanation of eclipses 7, 162, 172: moon borrows light from sun 138, 172, ii. 244: centrifugal force and centripetal tendency 172-3: geometry 170: tried to square circle 173, 220: on perspective 174: in *Erastus* 22, 174.
 Anaximander 67, 177: introduced *gnomon* 78, 139, 140: astronomy 139, ii. 244: distances of sun and moon 139: first map of inhabited earth *ib.*
 Anaximenes ii. 244.
 Anchor-ring, *see* Torē.
 Anderson, Alex., ii. 190.
 Angelo Poliziano ii. 26.
 Angle 'of a segment' and 'of a semicircle' 179: 'angle of contact' 178-9, ii. 202.
 Anharmonic property, of arcs of great circles ii. 269-70: of straight lines ii. 270, ii. 420-1.

cmius of Tralles 243, ii. 194,
 200-3, ii. 518, ii. 540, ii.
 1-3.
 phon 184, 219, 221-2, 224,
 1.
 tamba-Sūla-Sūtra 145-6.
 , F. F. 330.
 t, O. 181 n., 182.
 as 47.
 odorus, author of *Chronica*,
 .
 odorus ὁ λογιστικός: distich of,
 1, 133, 134, 144, 145.
 onius of Perga ii. 1, ii. 126.
 Arithmetic: *ἀκυρῶς* 234, ii.
 4, ii. 258 (approximation to
ib.), 'tetrads' 40, continued
 multiplications 54-7.
 Astronomy ii. 195-6: A. and
 cho Brahe 317, ii. 193: on
 cycles and eccentrics ii. 195-6,
 243: trigonometry ii. 253.
Conics ii. 126-75: text ii. 126-
 Arabic translations ii. 127,
 surfaces ii. 128-32, characteris-
 is ii. 132-3: conics obtained
 on oblique cone ii. 134-8,
 no property equivalent to
 ptosian equation (oblique axes)
 139, new names, *parabola*, &c.
 167, ii. 138, transformation
 coordinates ii. 141-7, tangents
 140-1, asymptotes ii. 148-9,
 tangles under segments of in-
 secting chords ii. 152-3, har-
 nic properties ii. 154-5, focal
 erties (central conics) ii. 156-
 normals as maxima and mini-
 ms ii. 159-67, construction of
 ms ii. 166-7, number of
 ms through point ii. 163-4,
 ositions giving evolutes ii.
 1-5.
 On contacts ii. 181-5 (lemmas
 ii. 416-17), three-circle pro-
 m ii. 182-5.
Section rationis ii. 175-9 (lemmas
 ii. 404-5).
Section spatii ii. 179-80, ii. 337,
 339.
Determinate section ii. 180-1
 mmas to, ii. 405-12).
 Comparison of dodecahedron
 icosahedron 419-20, ii. 192.
 Duplication of cube 262-3, ii.
 .

'General treatise' ii. 192-3, ii.
 253: on Book I of Euclid 358.
ρῶσις ii. 68, ii. 189-92 (lemmas
 to, ii. 412-16), rhombus-problem
 ii. 190-2, square-problem ii.
 412-18.
Plane Locii ii. 185-9 (lemmas to,
 ii. 417-19).
 On cochlias 232, ii. 193, 'sister
 of cochloid' 225, 231-2, On irra-
 tionals ii. 193, On the burning-
 mirror ii. 194, ii. 200-1.
 Application of arens 150-3: method
 attributed to Pythagoras 150,
 equivalent to solution of general
 quadratic 150-2, 394-6.
 Approximations to $\sqrt{2}$ (by means
 of 'side-' and 'diameter-' num-
 bers) 91-3, (Indian) 146: to $\sqrt{3}$
 (Ptolemy) 45, 62-3, (Archimedes)
 ii. 51-2: to π 232-5, ii. 194, ii.
 258: to surds (Heron) ii. 323-6,
 cf. ii. 547-9, ii. 553-4: to cube
 root (Heron) ii. 341-2.
 Apuleius of Madaura 97, 99.
 Archibald, R. C. 425 n.
 Archimedes 3, 52, 54, 180, 199, 202,
 203 n., 213, 217, 224-5, 229, 234,
 272, ii. 1.
 'Traditions ii. 16-17, engines ii.
 17, mechanics ii. 18, general
 estimate ii. 19-20.
 Works: character of, ii. 20-2,
 works extant ii. 22-3, lost ii. 23-
 5, 103; text ii. 25-7, MSS. ii. 26,
 editions ii. 27: *The Method* ii. 20,
 21, 22, 27-34, ii. 246, ii. 317-18:
On the Sphere and Cylinder ii. 34-
 50: *Measurement of a circle* ii. 50-
 6, ii. 253: *On Conoids and Sphe-
 roids* ii. 56-64: *On Spirals* 230-1,
 ii. 64-75 (cf. ii. 377-9), ii. 556-61:
Sand-reckoner ii. 81-5: *Quadra-
 ture of Parabola* ii. 85-91: me-
 chanical works, titles ii. 23-4,
Plane equilibria ii. 75-81: *On
 Floating Bodies* ii. 91-7, problem
 of crown ii. 92-4: *Liber assump-
 torum* ii. 101-3: Cattle-problem
 14, 15, ii. 28, ii. 97-8, ii. 447:
Catoptrica 444, ii. 24.
 Arithmetic: octads 40-1, frac-
 tions 42, value of π 232-3, 234,
 ii. 50-6: approximations to $\sqrt{3}$
 ii. 51-2.
 Astronomy ii. 17-18, sphero-

- making ii. 18, on Aristarchus's hypothesis ii. 3-4.
- Conics, propositions in, 438-9, ii. 122-6.
- Cubic equation solved by conics ii. 45-6.
- On Democritus 180, 327, equality of angles of incidence and reflection ii. 353-4, integral calculus anticipated ii. 41-2, 61, 62-3, 74, 89-90: Lemma or Axiom of A. 326-8, ii. 35: *vévςςςςςς* in, ii. 65-8 (Pappus on, ii. 68): on semi-regular solids ii. 98-101: triangle, area in terms of sides ii. 103: trisection of any angle 240-1.
- Archytas 2, 170, 212-16, ii. 1: on *μάθημα* 11, on *logistic* 14, on 1 as odd-even 71: on means 85, 86: no mean proportional between n and $n+1$, 90, 215: on music 214: mechanics 213: solution of problem of two mean proportionals 214, 219, 245, 246-9, 334, ii. 261.
- Argyryus, Isaac, 224 n., ii. 555.
- Aristaeus: comparison of five regular solids 420: *Solidi Loci* (conics) 438, ii. 116, 118-19.
- Aristaeus of Croton 86.
- Aristarchus of Samos 43, 139, ii. 1-15, ii. 251: date ii. 2: *σκάφη* of, ii. 1: anticipated Copernicus ii. 2-3: other hypotheses ii. 3, 4: treatise *On sizes and distances of Sun and Moon* ii. 1, 3, 4-15, trigonometrical purpose ii. 5: numbers in, 39: fractions in, 43.
- Aristonophus, vase of, 162.
- Aristophanes 48, 161, 220.
- Aristotelian treatise on indivisible lines 157, 346-8.
- Aristotherus 348.
- Aristotle 5, 120, 121: on origin of science 8: on mathematical subjects 16-17: on first principles, definitions, postulates, axioms 336-8.
- Arithmetic: reckoning by tens 26-7, why 1 is odd-even 71: 2 even and prime 73: on Pythagoreans and numbers 67-9: on the gnomon 77-8, 83.
- Astronomy: Pythagorean system 164-5, on hypothesis of concentric spheres 329, 335, ii. 244, on Plato's view about the earth 314-15.
- On the continuous and infinite 342-3: proof of incommensurability of diagonal 91: on principle of exhaustion 340: on Zeno's paradoxes 272, 275-7, 278-9, 282: on Hippocrates 22: encomium on Democritus 176.
- Geometry: illustrations from, 335, 336, 338-40, on parallels 339, proofs differing from Euclid's 338-9, propositions not in Euclid 340, on quadratures 184-5, 221, 223, 224 n., 271, on quadrature by lunes (Hippocrates) 184-5, 198-9: on Plato and regular solids 159: curves and solids in A. 341.
- Mechanics 344-6, 445-6: parallelogram of velocities 346: 'Aristotle's wheel' ii. 347-8.
- Aristoxenus 24 n., 66.
- Arithmetic (1) = theory of numbers (opp. to *λογιστική*) 13-16: early 'Elements of Arithmetic' 90, 216: systematic treatises, Nicomachus *Introd. Ar.* 97-112, Theon of Smyrna 112-3, Iamblichus, *Comm. on Nicomachus* 113-15, Dominicus ii. 538. (2) Practical arithmetic: originated with Phoenicians 120-1, in primary education 19-20.
- Arithmetic mean, defined 85.
- Arithmetica* of Diophantus 15-16, ii. 449-514.
- Arithmetical operations: see Addition, Subtraction, &c.
- Arrow of Zeno 276, 280-1.
- Āryabhaṭṭa 234.
- Asclepius of Tralles 99.
- Astronomy in elementary education 19: as secondary subject 20-1.
- Athelhard of Bath, first translator of Euclid 362-4.
- Athenaeus 144, 145.
- Athenaeus of Cyzicus 320-1.
- 'Attic' (or 'Herodianic') numerals 30-1.
- August, F. F. 299, 302, 361.
- Autolycus of Pitane 348: works *On the moving Sphere* 348-52, *On Risings and Settings* 352-3: relation to Euclid 251-2.
- Avverus, C. ii. 26.
- Axioms: Aristotle on, 336: = *Common Notions* in Euclid 370: Axioms of Archimedes 326-8, ii. 85.

- Babylonians: civilization of, 8, 9: system of numerals 28-9: sexagesimal fractions 29: 'perfect proportion' 86.
- Bachet, editor of Diophantus ii. 454-5, ii. 480.
- Bacon, Roger: on Euclid 367-8.
- Baillet, J. ii. 543.
- Baldi, B. ii. 308.
- Barlaam ii. 324 *n.*, ii. 554-5.
- Barocius ii. 545.
- Barrow, I., edition of Euclid, 869-70: on Book V 384.
- Bathycles 142.
- Bāudhāyana S. S. 146.
- Baynard, D. ii. 128.
- Benecke, A. 298, 302-3.
- Benedetti, G. B. 344, 446.
- Bertrand, J. ii. 324 *n.*
- Bessarion ii. 27.
- Besthorn, R. O. 362, ii. 310.
- Billingsley, Sir H. 369.
- Björnbo, A. A. 197 *n.*, 363, ii. 262.
- Blaas, C. 298.
- Blass, F. 182.
- Boeckh, A. 50, 78, 315.
- Boetius 87, 47, 90: translation of Euclid 359.
- Boissonade ii. 538.
- Bombelli, Rafael, ii. 454.
- Borchardt, L. 125, 127.
- Borelli, G. A. ii. 127.
- Bouillaud (Bullialdus) ii. 238, ii. 556.
- Braunmühl, A. von, ii. 268-9 *n.*, ii. 288, ii. 291.
- Breton (de Champ), P. 436, ii. 360.
- Bretschneider, C. A. 149, 183, 324-5, ii. 539.
- Brochard, V. 276-7, 279 *n.*, 282.
- Brougham, Lord, 436.
- Brugsch, H. K. 124.
- Bryson 210, 223-5.
- Burnet, J. 203 *n.*, 285, 314-15.
- Butcher, S. H. 299, 300.
- Buzengeiger ii. 324 *n.*
- Cajori, F. 283 *n.*
- Calculation, practical: the abacus 46-8, addition and subtraction 52, multiplication (i) Egyptian 52-3 (Russian ? 58 *n.*), (ii) Greek 58-8, division 58-60, extraction of square root 60-8, of cube root 63-4, ii. 341-2.
- Callimachus 141-2.
- Callippus: Great Year 177: system of concentric spheres 329, 335. ii. 244.
- Cambyses 5.
- Camerarius, Joachim, ii. 274.
- Canerer, J. G. ii. 360.
- Campanus, translator of Euclid 363-4.
- Canonic = theory of musical intervals 17.
- Cantor, G. 279.
- Cantor, M. 37-8, 123, 127, 131, 135, 182, ii. 203, ii. 207.
- Carpus of Antioch 225, 232, ii. 359.
- Case (πρώσις) 372, ii. 533.
- Cassini ii. 206.
- Casting out nines 115-17, ii. 549.
- Catoptric, theory of mirrors 18.
- Catoptrica: treatises by Euclid (?) 442, by Theon (?) 444, by Archimedes 444, and Heron 444, ii. 294, ii. 810, ii. 352-4.
- Cattle-problem of Archimedes 14, 15, ii. 23, ii. 97-8, ii. 447.
- Cavalieri, B. 180, ii. 20.
- Censorinus 177.
- Centre of gravity: definitions ii. 302, ii. 350-1, ii. 430.
- Certa Aristotelica ii. 531.
- Chalcidius ii. 242, 244.
- Chaldeans: measurement of angles by cells ii. 215-16: order of planets ii. 242.
- Charmandrus ii. 359.
- Chasles, M. ii. 19, 20: on Porismus 435-7, ii. 419.
- Chords, Tables of, 45, ii. 257, ii. 259-60.
- Chrysippus 179: definition of unit 69.
- Cicero 144, 359, ii. 17, 19.
- Circle: division into degrees ii. 214-15: squaring of, 173, 220-35, Antiphon 221-2, Bryson 223-4, by Archimedes's spiral 225, 230-1, Nicomedes, Dinostratus, and quadratrix 225-9, Apollonius 225, Carpus 225; approximations to π 124, 282-5, ii. 194, ii. 253, ii. 545.
- Cissoid of Diocles 264-6.
- Clauser, Th. 200.
- Cleantes ii. 2.
- Cleomedes: 'paradoxical' eclipse 6: *De motu circulari* ii. 235-8, 244.
- Cleonides 444.
- Cochlias 232, ii. 193.

- Cochloids* 238-40: 'sister of cochloid' 225, 231-2.
- Coins and weights, notation for, 31.
- Columella* ii. 303.
- Commandinus, F., translator of Euclid, 365, 425, Apollonius ii. 127, *Analemma* of Ptolemy ii. 287, *Planisphaerium* ii. 292, Heron's *Pneumatica* ii. 308, Pappus ii. 360, Serenus ii. 519.
- Conchoid of Nicomedes 233-40.
- Conclusion* 370, ii. 533.
- Cones: Democritus on, 179-80, ii. 110: volume of, 176, 180, 217, 327, 413, ii. 21, ii. 332: volume of frustum ii. 334: division of frustum in given ratio ii. 340-3.
- Conic sections: discovered by Menaechnus 252-3, ii. 110-16: Euclid's *Conics* and Aristaeus's *Solid Loci* 438, ii. 116-19: propositions included in Euclid's *Conics* ii. 121-2 (focus-directrix property 243-4, ii. 119-21), conics in Archimedes ii. 122-6: names due to Apollonius 150, ii. 138: Apollonius's *Conics* ii. 126-75: conics in *Fragmentum Bobiense* ii. 200-203: in Anthemius ii. 541-3.
- Conon of Samos ii. 16, ii. 359.
- Construction* 370, ii. 533.
- Conversion of ratio (convertendo)* 386.
- Cook-Wilson, J. 300 n., ii. 370.
- Counter-earth 164.
- Croesus 4, 129.
- Ctesibius 213: relation to Philon and Heron ii. 298-302.
- Cube: called 'geometrical harmony' (Philolaus) 85-6.
- Cube, duplication of: history of problem 244-6: reduction by Hippocrates to problem of two mean proportionals 2, 183, 200, 245: solutions, by Archytas 246-9, Eudoxus 249-51, Menaechnus 251-6, 'Plato' 255-8, Eratosthenes 258-60, Nicomedes 260-2, Apollonius, Philon, Heron 262-4, Dioctes 264-6, Sporus and Pappus 266-8: approximation by plane method 268-70.
- Cube root, extraction of, 63-4: Heron's case ii. 341-2.
- Cubic equations, solved by conics, 237-8, ii. 45-6, ii. 46: particular case in Diophantus ii. 465, ii. 512.
- Curtze, M. 75 n., ii. 309.
- Cyrus 129.
- Dactylus*, 1/24th of ell, ii. 216.
- Damastes of Sigeum 177.
- Damianus ii. 294.
- Darius-vase 48-9.
- D'Armagnac, G. ii. 26.
- Dasypodius ii. 554 n.
- De la Hire ii. 550.
- De levi et ponderoso* 445-6.
- Decagon inscribed in circle, side of, 416: area of, ii. 328.
- Dec, John, 369, 425.
- Definitions: Pythagorean 166: in Plato 289, 292-4: Aristotle on, 337: in Euclid 373: *Definitions* of Heron, ii. 314-16.
- Demetrius of Alexandria ii. 260, ii. 359.
- Democritus of Abdera 12, 119, 121, 182: date 176, travels 177: Aristotlo's encomium 176: list of works (1) astronomical 177, (2) mathematical 178: on irrational lines and solids 156-7, 181: on angle of contact 178-9: on circular sections of cone 179-80, ii. 110: first discovered volume of cone and pyramid 176, 180, 217, ii. 21: atoms mathematically divisible *ad inf.* 181: 'Εκκρίσματα 178, 181: on perspective 174: on Great Year 177.
- Dercyllides ii. 244.
- Descartes 75 n., 279.
- Dicenearchus ii. 242.
- Dichotomy* of Zeno 275, 278-80.
- Diels, H., 142 n., 176, 178, 184, 188.
- Digamma: from Phoenician Vau 32: signs for, *ib.*
- Digit* 27.
- Dinostratus 225, 229, 320-1, ii. 359.
- Diocles: inventor of cissoid 264-6: solution of Archimedes *On Sph. and Cyl.* II. 4, ii. 47-8: on burning-mirrors ii. 200-3.
- Diodorus (math.): on parallel-postulate 358: *Analemma* of, ii. 287, ii. 359.
- Diodorus Siculus 121, 141, 142, 176.
- Diogenes Laertius 144, 145, 177, 201.
- Dionysius, Plato's master, 22.
- Dionysius, a friend of Heron, ii. 800.
- Dionysodorus ii. 46, ii. 218-10, ii. 334-5.
- Diophantus of Alexandria: date ii.

448: works and editions ii. 448-55: *Arithmetica* 15-16: fractions in, 42-4: notation and definitions ii. 455-61: signs for unknown (x) and powers ii. 456-9, for *minus* ii. 459: methods ii. 462-79: determinate equations ii. 462-5, 484-90: indeterminate analysis ii. 466-76, 491-514: 'Porisms' ii. 449, 450, 451, ii. 479-80: propositions in theory of numbers ii. 481-4: conspectus of *Arithmetica* ii. 484-514: *On Polygonal Numbers* 16, 84, ii. 514-17: '*Moriastica*' ii. 449.

Dioptra 18, ii. 256: Heron's *Dioptra* ii. 345-6.

division: Egyptian method 58, Greek 58-60: example with sexagesimal fractions (Theon of Alexandria) 59-60.

Divisions (of Figures), *On*, by Euclid 425-80: similar problems in Heron ii. 336-40.

Dodecagon, area of, ii. 328.

Dodecahedron: discovery attributed to Pythagoras or Pythagoreans 65, 141, 158-60, 162: early occurrence 160: inscribed in sphere (Euclid) 418-19, (I'Appus) ii. 360: Apollonius on, 419-20: volume of, ii. 835.

Dominius ii. 538.

Dositheus ii. 84.

Duhem, P. 446.

Dupuis, J. ii. 239.

Earth: measurements of, ii. 82, (Eratosthenes) ii. 106-7, (Posidonius) ii. 220.

Ecliptic: obliquity discovered by Oenopides 174, ii. 244: estimate of inclination (Eratosthenes, Ptolemy) ii. 107-8.

Elephantus 317, ii. 2.

Edfu, Temple of Horus 124.

Egypt: priests 4-5, 8-9: relations with Greece 8; origin of geometry in, 120-2: orientation of temples 122.

Egyptian mathematics: numeral system 27-8, fractions 28, multiplication, &c. 14-15, 52-3: geometry (mensuration) 122-8: triangle (3, 4, 5) right-angled 122, 147: value of π 124, 125: measure-

ment of pyramids 126-8: maps (regional) 139: algebra in Pappus Rhind, &c. ii. 440-1.

Eisenlohr, A. 123, 126, 127.

Eisenmann, H. J. ii. 360.

Elements: as known to Pythagoreans 166-8: progress in, down to Plato 170-1, 175-6, 201-2, 209-13, 216-17: writers of Elements, Hippocrates of Chios 170-1, 201-2, Leon, Theudius 320-1: other contributors to, I'codamas, Archytas 170, 212-13, Theaetetus 269-12, 354, Hermotimus of Colophon 320, Eudoxus 320, 323-9, 354: *Elements* of Euclid 357-419: the so-called 'Books XIV, XV' 419-21.

Ell, as measure of angles ii. 215-16.

Empedocles: on Pythagoras 65.

Eneström, G. ii. 341-2.

Enneagon: Heron's measurement of side ii. 259, of area ii. 328-9.

Epanthema of Thymaridas (system of simple equations) 94: other types reduced to, 94-6.

Equations: simple, in Pappus Rhind, &c. ii. 441: in *eplanthema* of Thymaridas and in Iamblichus 94-6: in Greek anthology ii. 441-3: indeterminate, *see* Indeterminate Analysis: *see also* Quadratic, Cubic.

Eratosthenes ii. 1, 16: date, &c. ii. 104: *stere* (στάχυς) for finding primes 16, 100, ii. 105: on duplication of cube 244-6, 251, 258-60: the *Platonics* ii. 104-5: *On Means* ii. 105-6, ii. 359: *Measurement of earth* ii. 106-7, ii. 242, ii. 346: astronomy ii. 107-9: chronology and *Geographica* ii. 109: on *Octaeteris* *ib.*

Erycinus ii. 359, 365-8.

Euclid 2-3, 93, 131: date, &c. 354-6: stories of, 25, 354, 357: relation to predecessors 354, 357: Pappus on, 356-7.

Arithmetic: classification and definitions of numbers 72-3, 397, 'perfect' numbers 74, 402: formula for right-angled triangles in rational numbers 81-2, 405.

Conics 433-9, ii. 121-2, focus-directrix property ii. 119-21: on ellipse 439, ii. 111, ii. 125.

- Data* 421-5, *Divisions* (of figures) 425-30, ii. 336, 339.
- Elements*: text 360-1, Theon's edition 358, 360, ii. 527-8, translation by Boëtius 359, Arabic translations 362, ancient commentaries 358-9, *editio princeps* of Greek text 360, Greek texts of Gregory, Peyrard, August, Heiberg 360-1: Latin translations, Athelhard 362-3, Gherard 363. Campanus 363-4, Commandinus 365: first printed editions, Radolt 364-5, Zamberti 365: first introduction into England 363: first English editions, Billingsley, &c. 369-70: Euclid in Middle Ages 365-9, at Universities 368-9: analysis of, 373-419: arrangement of postulates and axioms 361: I. 47, how originally proved 147-9: parallel-postulate 358, 376, ii. 227-30, ii. 295-7, ii. 534: so-called 'Books XIV, XV' 419-21.
- Mechanics* 445-6: *Music* 444-5, *Sectio canonis* 17, 90, 215, 444-5: *Optics* 17-18, 441-4: *Phaenomena* 849, 351-2, 440-1, ii. 249: *Porisms* 431-8, lemmas to, ii. 419-24: *Pseudaria* 430-1: *Sursum-Loci* 243-4, 439-40, lemmas to, ii. 119-21, ii. 425-6.
- Eudemus 201, 209, 222: *History of Geometry* 118, 119, 120, 130, 131, 135, 150, 171: on Hippocrates's lunes 173, 182, 183-98: *History of Astronomy* 174, 329, ii. 244.
- Eudoxus 24, 118, 119, 121, 320, 322-4: new theory of proportion (that of Eucl. V. ii) 2, 153, 216, 325-7: discovered method of exhaustion 2, 176, 202, 206, 217, 222, 326, 327-9: problem of two mean proportionals 245, 246, 249-51: discovered three new means 56: 'general theorems' 323-4: *On speeds*, theory of concentric spheres 329-34, ii. 244: *Phaenomena* and *Mirror* 322.
- Eugenius Siculus, Admiral, ii. 293.
- Euler, L. 75 n., ii. 482, ii. 483.
- Euphorbus (= Pythagoras) 142.
- Eurytus 69.
- Eutocius 52, 57-8, ii. 25, ii. 45, ii. 126, ii. 518, ii. 540-1.
- Exhaustion, method of, 2, 176, 202, 217, 222, 326, 327-9: development of, by Archimedes 224, ii. 35-6.
- False hypothesis: Egyptian use ii. 441: in Diophantus ii. 488, 489.
- Fermat, P. 75 n., ii. 20, ii. 185, ii. 454, ii. 480, ii. 481-4: on Porisms 485.
- Fontenelle ii. 556.
- Fractions: Egyptian (submultiples except $\frac{1}{2}$) 27-8, 41: Greek systems 42-4: Greek notation *ib.*: sexagesimal fractions, Babylonian 29, in Greek 44-5.
- 'Friendly' numbers 75.
- Galilei 344, 446.
- Geponicus, Liber*, 124, ii. 309, ii. 318, ii. 344.
- Geminus 119, ii. 222-34: on arithmetic and logistic 14: on divisions of optics, &c. 17-18: on original steps in proof of Eucl. I 32, 135-6: on parallels 358: attempt to prove parallel-postulate ii. 227-30: on original way of producing the three conics ii. 111: encyclopaedic work on mathematics ii. 223-31: on Posidonius's *Meteorologica* ii. 231-2: *Introduction to Phaenomena* ii. 232-4.
- Geodesy (*γεωδαια*) = mensuration (as distinct from geometry) 16-17.
- Geometric mean, defined (Archylas) 85: one mean between two squares (or similar numbers), two between cubes (or similar solid numbers) 89-90, 112, 201, 297, 400: no rational mean between consecutive numbers 90, 215.
- 'Geometrical harmony' (Philolaus's name for cube) 85-6.
- Geometry: origin in Egypt 120-2: geometry in secondary education 20-1.
- Georgius Pachymeres ii. 453, ii. 546.
- Gerbert (Pope Sylvester II) 865-7: geometry of, 366: ii. 547.
- Gerhardt, C. J. ii. 360, ii. 547.
- Gherard of Cremona, translator of Euclid and an-Nairizi 363, 367, ii. 309: of Menelaus ii. 252, ii. 262.

Ghetaaldi, Marino, ii. 190.
 Girard, Albert, 435, ii. 455.
Gnomon: history of term 78-9:
 gnomons of square numbers 77-
 8, of oblong numbers 82-3, of
 polygonal numbers 79: in appli-
 cation of areas 151-2: use by
 al-Karkhī 109-10: in Euclid 379:
 sun-dial with vertical needle 139.
 Gomperz, Th. 176.
 Govi, G. ii. 293 n.
 Gow, J. 88.
 Great Year, of Oenopides 174-5,
 of Callippus and Democritus 177.
 Gregory, D. 360-1, 440, 441, ii. 127.
 Griffith, F. Ll. 125.
 Günther, S. ii. 325 n., ii. 550.
 Guldin's theorem, anticipated by
 Pappus ii. 403.
Halicarnassus inscriptions 32-3,
 34.
 Halley, E., editions of Apollonius's
Conics ii. 127-8, and *Sectio ratio-
 nis* ii. 175, 179, of Menelaus ii.
 252, ii. 262, of extracts from
 Pappus ii. 360, of Serenus ii. 519.
 Halma, editor of Ptolemy ii. 274,
 275.
 Hammer-Jensen, I. ii. 300 n., ii.
 304 n.
 Hankel, H. 145, 149, 288, 369, ii.
 483.
 Hardy, G. H. 280.
 Harmonic mean (originally 'sub-
 contrary') 85.
Harpodonaptæ, 'rope-stretchers'
 121-2, 178.
 Hārūn ar-Rashīd 362.
Hau-calculations (Egyptian) ii.
 440-1.
 Hecataeus of Miletus 65, 177.
 Heibsen, J. L. 238 n.
 Heiberg, J. L. 184, 187 n., 188,
 192 n., 196-7 n., 315, 361, ii. 203,
 ii. 309, 310, 316, 318, 319, ii. 519,
 ii. 535, 543, 553, 555 n.
Helicoph 111.
 Hendeceagon in a circle (Heron) ii.
 259, ii. 329.
 Henry, C. ii. 459.
 Heptagon in a circle, ii. 103:
 Heron's measurement of, ii. 328.
 Heraclides of Pontus 24, ii. 281-2:
 discovered rotation of earth about
 axis 310-17, ii. 2-3, and the Venus

and Mercury revolve about sun
 312, 317, ii. 2, ii. 244.
 Heraclitus of Ephesus 65.
 Heracitus, mathematician ii. 192,
 ii. 359, ii. 412.
 Hermannus Secundus ii. 292.
 Hermesianax 142 n., 163.
 Hermodorus ii. 359.
 Hermotimus of Colophon 320-1:
 Elements and Loci *ib.*, 354.
 'Herodianic' (or 'Attic') numerals
 30-1.
 Herodotus 4, 5, 48, 65, 121, 139.
 Heron of Alexandria 121, ii. 198,
 ii. 259: controversies on date ii.
 298-307: relation to Ctesibius
 and Philon ii. 298-302, to Pappus
 ii. 299-300, to Posidonius and
 Vitruvius ii. 302-3, to *agrimen-
 sores* ii. 303, to Ptolemy ii. 303-6.
 Arithmetic: fractions 42-4, mul-
 tiplications 58, approximation to
 surds ii. 51, ii. 323-6, approxima-
 tion to cube root 64, ii. 341-2,
 quadratic equations ii. 344, in-
 determinate problems ii. 344,
 444-7.
 Character of works ii. 307-8:
 list of treatises ii. 308-10.
 Geometry ii. 310-14, *Definitions*
 ii. 314-16: comm. on Euclid's
Elements 358, ii. 310-14: proof of
 formula for area of triangle in
 terms of sides ii. 321-3: duplica-
 tion of cube 262-3.
Metrica ii. 320-43: (1) mensu-
 ration ii. 316-35: triangles ii.
 320-3, quadrilaterals ii. 326,
 regular polygons ii. 326-9, circle
 and segments ii. 329-31: volumes
 ii. 331-5, *Stylaxos* ii. 332-3, frus-
 tum of cone, sphere and segment
 ii. 334, *tore* ii. 334-5, five regular
 solids ii. 335. (2) divisions of
 figures ii. 336-48, of frustum of
 cone ii. 342-3.
Mechanics ii. 346-52: on Ar-
 chimedes's mechanical works ii.
 23-4, on centre of gravity ii. 350-1,
 352.
Belopoeica 18, ii. 308-9, *Catop-
 trica* 18, ii. 294, ii. 310, ii. 352-4.
Dioptra ii. 345-6, *Pneumatica*
 and *Automata* 18, ii. 308, 310.
 On *Water-clocks* ii. 429, ii. 536.
 Heron, teacher of Proclus ii. 520.

- 'Heron the Younger' ii. 545.
 Heronns 99.
 Hiectas 317.
 Hierius 268, ii. 359.
 Hieronymus 129.
 Hilāl b. Abi Hilāl al-Himṣī ii. 127.
 Hiller, E. ii. 239.
 Hilprecht, H. V. 29.
 Hipparchus ii. 3, 18, 198, 216, 218:
 date, &c. ii. 253: work ii. 254-6:
 on epicycles and eccentrics ii.
 243, ii. 255: discovery of preces-
 sion ii. 254: on mean lunar month
 ii. 254-5: catalogue of stars ii.
 255: geography ii. 256: trigono-
 metry ii. 257-60, ii. 270.
 Hippasus 65, 85, 80, 214: construc-
 tion of 'twelve pentagons in a
 sphere' 160.
 Hippias of Elis: taught mathe-
 matics 28: varied accomplish-
 ments *ib.*, lectures in Sparta 24:
 inventor of *quadratrix* 2, 171, 182,
 210, 225-6.
 Hippocrates of Chios 2, 182, 211:
 taught for money 22: first writer
 of *Elements* 119, 170, 171: ele-
 ments as known to, 201-2:
 assumes *vrivis* equivalent to solu-
 tion of quadratic equation 88,
 195-6: on quadratures of lunes
 170, 171, 173, 182, 183-99, 220,
 221: proved theorem of Eucl. XII
 2, 187, 328: reduced duplication
 of cube to problem of finding
 two mean proportionals 2, 183,
 200, 245.
 Hippolytus: on *πυλῖνες* (bases) and
 'rule of nine' and 'seven' 115-16.
 Hippopede of Eudoxus 333-4.
 Homer 5.
 'Horizon': use in technical sense by
 Euclid 352.
 Horsley, Samuel, ii. 190, ii. 360.
 Hultsch, F. 204, 230, 349, 350, ii. 51,
 ii. 308, ii. 318, 319, ii. 361.
 Hunrath, K. ii. 51.
 Hunt, A. S. 142.
 Hypatia ii. 449, ii. 519, ii. 528-9.
 Hypotenuse, theorem of square on,
 142, 144-9: Proclus on discovery
 of, 145: supposed Indian origin
 145-6.
 Hypsicles: author of so-called Book
 XIV of Eucl. 419-20, ii. 192: de-
 finition of 'polygonal number' 84,
 ii. 213, ii. 515: 'Αραφορικὸς ii.
 213-18, first Greek division of
 zodiac circle into 360 parts ii. 214.
 Iamblichus 4, 69, 72, 73, 74, 75, 86,
 107, ii. 515, 529: on *ἐπὶ ἀριθμῶν* of
 Thymaridas, &c. 94-6: works
 113-14: comm. on Nicomachus
 113-15: squares and oblong num-
 bers as 'race-courses' 114: pro-
 perty of sum of numbers $3n-2$,
 $3n-1$, $3n$ 114-15.
 Ibn al-Haitham, on burning-mirrors
 ii. 201: ii. 453.
 Icosahedron 159: discovery attrib-
 uted to Theaetetus 162: volume
 of, ii. 335.
 Incommensurable, discovery of, 65,
 90-1, 154: proof of incommensu-
 rability of diagonal of square 91.
 Indeterminate analysis: first cases,
 right-angled triangles in rational
 numbers 80, 81, 'side-' and 'dia-
 meter-' numbers 91-3, ii. 536:
 rectangles with area and peri-
 meter numerically equal 96-7:
 indeterminate equations, first
 degree ii. 443, second degree ii.
 443-4 (see also Diophantus), in
 Heronian collections ii. 344, ii.
 444-7.
 India: rational right-angled tri-
 angles in, 145-6: approximation
 to $\sqrt{2}$, 146.
 Indian Table of Sines ii. 253.
 Irrational: discovered by Pythago-
 reans 65, 90-1, 154, and with
 reference to $\sqrt{2}$, 155, 168: Demo-
 critus on, 156-7, 181: Theodorus on,
 203-9: extensions by Theaetetus
 209-12, Euclid 402-11, Apollonius
 ii. 193.
 Isaac Argyrus 224 *n.*, ii. 555.
 Ishāq b. Hunain, translator of
 Euclid 362, of Menelaus ii. 261,
 and Ptolemy ii. 274.
 Isidorus Hispalensis 305.
 Isidorus of Miletus 421, ii. 25, ii.
 518, ii. 540.
 Isocrates: on mathematics in edu-
 cation 21.
 Isoperimetric figures ii. 206-13, ii.
 390-4.
 Jacob b. Machir ii. 252, ii. 262.
 Jacobus Cremonensis ii. 26-7.

Jan, O. 444.
 Joachim Camorarius ii. 274.
 Joachim, H. H. 348 n.
 Johannes de Sacrobosco 368.
 Jordanus Nemorarius ii. 328.
 Jourdain, P. E. B. 283 n.
 Kahun Papyri 125, 126.
 Kunt 173.
 Keil, B. 84-5.
 Kepler ii. 20, ii. 99.
 Köchly, H. A. T. ii. 309.
 Koppa (Q for 90) = Phoenician Qoph 32.
 Kubitschek, W. 50.
 Lagrange ii. 483.
 Laird, A. G. 306 n.
 Laplace 173.
 Larfeld, W. 31 n., 33-4.
 Lawson 436.
 Leibniz 279, ii. 20.
 Lemma 373, ii. 538.
 Leodunus of Thasos 120, 170, 212, 291, 319.
 Leon 319.
 Leon (of Constantinople) ii. 25.
 Leonardo of Pisa 367, 426, ii. 547.
 Lepsius, G. R. 124.
 Leucippus 181.
 Libri, G. ii. 556.
 'Linear' (of numbers) 73.
 'Linear' loci and problems 218-19.
 Lines, classification of, ii. 226.
 Livy ii. 18.
 Loci: classification of, 218-19, plano, solid, linear 218: loci on surfaces 219: 'solid loci' ii. 116-19.
 Loftus, W. K. 28.
 Logistic (opp. to 'arithmetica'), science of calculation 13-16, 23, 58.
 Logistica speciosa and numerosa (Viola) ii. 456.
 Lorin, G. iv-v, 350 n., ii. 293 n.
 Luca Paciolo 367, ii. 324 n.
 Lucas, B. 75 n.
 Lucian 75 n., 77, 99, 161, ii. 18.
 Lucretius 177.
 Magic squares ii. 550.
 Magnus, Logistica 234-5.
 Mamercus or Mamortius 140, 141, 171.
 al-Ma'mūn, Caliph 362.
 al-Mansūr, Caliph 362.

Manus, for number 27.
 Marinus 444, ii. 192, ii. 537-8.
 Martianus Capella 359, 365.
 Martin, T. H. ii. 238, ii. 546.
 Maslama b. Ahmad al-Majrī ii. 292.
 Massalia 8.
 Mastaba tombs 123.
 Mathematics: meaning 10-11, classification of subjects 11-13: branches of applied mathematics 17-18: mathematics in Greek education 18-25.
 Maurolycus ii. 262.
 Means: arithmetic, geometric, and subcontrary (harmonic) known in Pythagoras's time 85: defined by Archytas *ib.*: fourth, fifth, and sixth discovered, perhaps by Eudoxus 86, four more by Myonides and Euphranor 86: ten means in Nicomachus and Pappus 87-9, Pappus's propositions 88-9: no rational geom. mean between successive numbers (Archytas) 90, 215.
 Mechanics, divisions of, 18: writers on, Archytas 213, Aristotle 344-6, 445-6, Archimedes ii. 18, ii. 23-4, ii. 75-81, Ptolemy ii. 295, Heron ii. 346-52, Pappus ii. 427-34.
 Megethion ii. 360.
 Memus, Johannes Baptista, ii. 127.
 Menaechnus 2, 25, 251-2, 320-1: discoverer of conic sections 251-8, ii. 110-16: solved problem of two mean proportionals 245, 246, 251-5: on 'problems' 318.
 Menelaus of Alexandria ii. 198, ii. 252-3: date, &c. ii. 260-1: Table of Chords ii. 257: *Sphaerica* ii. 261-73: Menelaus's theorem ii. 266-8, 270: anharmonic property ii. 269: *παράδοξος κύρτος* ii. 260-1.
 Mensa Pythagorica 47.
 Mensuration: in primary education 19: in Egypt 122-8: in Heron ii. 316-85.
 Moton 220.
 Metrodorus ii. 442.
 Minus, sign for, in Diophantus ii. 459-60.
 Mochus 4.
 Moschopoulos, Manuel, ii. 549-50.
 Muhammad Bagdadinus 425.
 Multiplication: Egyptian method

- 52-3, Greek 53-4, 'Russian' 53 *n.*: examples from Eutocius, Heron, Theon 57-8: Apollonius's continued multiplications 54-7.
- Multiplication Table 53.
- Moran*, an angular measure ii. 215.
- Musical intervals and numerical ratios 69, 75-6, 85, 165.
- Myriads, 'first', 'second', &c., notation for, 39-40.
- Nagl, A. 50.
- an-Nairizi: *comm.* on Euclid 363, ii. 224, ii. 228-30, ii. 309-10.
- Nasiraddin at-Tusi: version of Euclid 362, of Apollonius's *Conics* ii. 127: of Ptolemy ii. 275.
- Naucratid inscriptions 33.
- Nemesius 441.
- Neocides 319.
- Ner* (Babylonian) (= 600) 28, ii. 215.
- Nesselmann, G. H. F. ii. 450-1, ii. 455-6.
- Newton 370, ii. 20, ii. 182.
- Nicolas Rhadgas 40, ii. 324 *n.*, ii. 550-8.
- Nicomachus of Gerasa 12, 69, 70, 72, 73, 74, 76, 83, 85, 86, ii. 233, ii. 515: works of, 97: *Introductio arithmetica*: character of treatise 98-9, contents 99-112, classification of numbers 99-100: on 'perfect' numbers 74, 100-1: on ten means 87: on a 'Platonic' theorem 297: sum of series of natural cubes 109-10.
- Nicomedes 225-6, ii. 199: cochloids or conchoids 238-40: duplication of cube 260-2.
- Niloxenus 129.
- Nine, rule of, 115-16: casting out nines ii. 549.
- Nipkus, M. Junius, 132.
- Nix, L. ii. 128, 131, ii. 309.
- Noël, G. 282.
- Number: defined, by Thales 69, by Moderatus, Eudoxus, Nicomachus, Aristotle 70: classification of numbers 70-4: 'perfect', 'over-perfect' and 'defective' numbers 74-5, 'friendly' 75, figured 76-9: 'oblong', 'prolate' 82-3, 108, 114, similar plane and solid numbers 81-2, 90, solid numbers classified 106-8: 'the number in the heaven' (Pythagorean) 68, 'number' of an object 69.
- Numerals: systems of, decimal, quinary, vigesimal 26: origin of decimal system 26-7: Egyptian 27-8; Babylonian systems (1) decimal 28, (2) sexagesimal 28-9: Greek (1) 'Attic' or 'Herodianic' 30-1: (2) alphabetic system, original in Greece 31-7, how evolved 31-2, date of introduction 33-5, mode of writing 36-7, comparison of two systems 37-9: notation for large numbers, Apollonius's tetrads 40, Archimedes's octads 40-1.
- Nymphodorus 213.
- 'Oblong' numbers 82-3, 108, 114: gnomons of, 82-3.
- Ocreatus, 111.
- Octads, of Archimedes 40-1.
- Octagon, regular, area of, ii. 328.
- Octahedron 159, 160, 162: volume of, ii. 335.
- 'Odd' number defined 70-1: 1 called 'odd-even' 71: 'odd-even', 'odd-times-odd', &c., numbers 71-4.
- Oenopides of Chios 22, 121: discovered obliquity of ecliptic 138, 174, ii. 244: Great Year of, 174-5: called perpendicular *gnomon-wise* 78, 175: two propositions in elementary geometry 175.
- Olympiodorus 444.
- One, the principle of number 69.
- Oppermann ii. 324 *n.*
- Optics: divisions of, 17-18: of Euclid 441-4: of Ptolemy ii. 293-4.
- Oval of Cassini ii. 206.
- Oxyrhynchus Papyri 142.
- Pamphile, 131, 133, 134.
- Pandrosion ii. 360.
- Pappus (*see also* Table of Contents, under Chap. XIX) ii. 17-18, ii. 175, 180, 181, 182, 183, 185, 186, 187, 188, 189, 190, ii. 207, 211, 212, 213, ii. 262, ii. 337, ii. 355-439: on Apollonius's tetrads 40, on Apollonius's continued multiplications 54-7: on ten means 87-9: on mechanical works of Archimedes ii. 23-4: on conics

- of Euclid and Apollonius 438, proof of focus-directrix property ii. 120-1: commentary on Euclid 358, ii. 356-7, on Book X 154-5, 209, 211, ii. 198: commentary on Euclid's *Data* 421-2, ii. 357, on Diodorus's *Analemma* ii. 287, scholia on *Syntaxis* ii. 274: on classification of problems and loci (plane, solid, linear) 218-19, ii. 117-18, criticism on Archimedes and Apollonius 288, ii. 68, ii. 167: on surface-loci 439-40, ii. 425-6: on Euclid's *Parisms* 491-3, 486-7, ii. 270, ii. 419-24: on 'Treasury of Analysis' 421, 422, 439, ii. 899-427: on *cochloids* 288-9: on *quadratrix* 229-30, ii. 379-80, constructions for, ii. 880-2: on duplication of cube 266-8, 268-70: on trisection of any angle 241-3, ii. 885-6, *reūs* with regard to parallelogram 236-7: on isoperimetry (cf. Zenodorus) ii. 207, ii. 211-12, ii. 390-4.
- 'Paradoxes' of Erycinus ii. 865-8.
- Parallelogram of velocities 846, ii. 848-9.
- Parapegma* of Democritus 177.
- Parmenides 138.
- Paterius ii. 536-7.
- Patricius ii. 313, 319.
- Pebbles, for calculation 46, 48.
- Pentagon, regular: construction Pythagorean 160-2: area of, ii. 827.
- Pentagram, Pythagorean 161-2 (see *Errata*).
- 'Perfect' numbers 74-5: list of first ten *ib.*: contrasted with 'over-perfect' and 'defective' *ib.*: 10 with Pythagoreans 75.
- 'Perfect' proportion 86.
- Pericles 172.
- Pericles, a mathematician ii. 800.
- Perseus 226: spiric sections ii. 203-6.
- 'Phaenomena' = observational astronomy 17: 822, 349.
- Philippus of Opus 354: works by, 321: on polygonal numbers 84, ii. 515: astronomy 321.
- Philolus 67, 72, 76, 78, 86, 158, ii. 1: on odd, even, and even-odd numbers 70-1: Pythagorean non-geocentric astronomy attributed to, 163-4.
- Philon of Byzantium 213: duplication of cube 262-3: Philon, Ctesibius and Heron ii. 298-302.
- Philon of Gadara 234.
- Philon of Tyana ii. 260.
- Philoponus, Joannes, 99, 223, 224 *n.*
- Phocceans 7.
- Phocus of Samos 138.
- Phoenician alphabet, how treated by Greeks 31-2: arithmetic originated with Phoenicians 120-1.
- 'Piremus' or 'peremus' in pyramid 126, 127.
- 'Plane' loci 218.
- 'Plane' problems 218-19.
- Planisphaerium* of Ptolemy ii. 292-3.
- Planudes, Maximus, 117, ii. 458, ii. 546-9.
- Plato 19, 22, 24, 121, 142 *n.*, 170, 176: *Θεοδίδει γινώσκειν* 10: *μυθολογία* 24, 355: on education in mathematics 19-20, 284: on mathematical 'arts', measurement and weighing 308, instruments for, 308-9, principle of lever 309: on optics 309, 441: on music 310: Plato's astronomy 310-15: on arithmetic and logistic 13-14: classification of numbers, odd, even, &c. 71-2, 292: on number 5040, 294: the Geometrical Number, 305-8: on arithmetical problems 15, ii. 442: on geometry 286-8, constructions alien to true geometry *ib.*: ontology of mathematics 288-9: hypotheses of mathematics 289-90: two intellectual methods 290-2: supposed discovery of mathematical analysis, 120, 212-13, 291-2: definitions of various species of numbers 292, figure 292-3, line and straight line 293, circle and sphere 293-4: on points and indivisible lines 293: formula for rational right-angled triangles 81, 304: 'rational' and 'irrational' diameter of $\sqrt{5}$ 93, 308-7: Plato and the irrational 156, 203-5, 304: on solid geometry 12-13, 303: on regular and semi-regular solids 294-7: Plato and duplication of cube 245-6, 255, 287-8, 303: on geometric means between two squares and two cubes respectively 80, 112,

proportion 294: two geometrical passages in *Meno* 297-303: propositions 'on the section' 304, 324-5.
 'Platonic' figures (the regular solids) 158, 162, 294-5, 296-7.
 Playfair, John, 436.
 Pliny 129, ii. 207.
 Plutarch 84, 96, 128, 129, 130, 133, 144, 145, 167, 179, ii. 2, 3, ii. 516: on Archimedes ii. 17-18.
 Point: defined as a 'unit having position' 69, 166: Plato on points 293.
 Polybius 48, ii. 17 n., ii. 207.
 Polygon: propositions about sum of exterior or interior angles 144: measurement of regular polygons ii. 326-9.
 Polygonal numbers 15, 76, 79, ii. 213, ii. 514-17.
 Polyhedra, *see* Solids.
 Porism (1) = corollary 372: (2) a certain type of proposition 373, 431-8: *Porisms* of Euclid, *see* Euclid: of Diophantus, *see* Diophantus.
 Porphyry 145: commentary on Euclid's *Elements* 358, ii. 529.
 Poselger, F. T. ii. 455.
 Posidonius ii. 219-22: definitions ii. 221, 226; on parallels 358, ii. 228: *versus* Zeno of Sidon ii. 221-2: *Meteorologica* ii. 219: measurement of earth ii. 220: on size of sun ii. 108, ii. 220-1.
 Postulates: Aristotle on, 336: in Euclid 336, 374-5: in Archimedes 336, ii. 75.
 Powers, R. E. 75 n.
 Prestet, Jean, 75 n.
 Prime numbers and numbers prime to one another 72-3: defined 73: 2 prime with Euclid and Aristotle, not Theon of Smyrna and Neo-Pythagoreans *ib.*
 Problems: classification 218-19: plane and solid ii. 117-18: problems and theorems 318, 431, ii. 533.
 Proclus 12, 99, 175, 183, 213, 224 n., ii. 529-37: *Comm. on Eucl. I.* ii. 530-5: sources ii. 530-2: 'summary' 118-21, 170, object of, 170-

I. 47, 145, 147: attempt parallel-postulate 358, i. loci 219: on porisms 431: Euclid's music 444: *Republic* 92-3, ii. 536-7: *posis of astronomical* ii. 535-6.
 Prodicus, on secondary 20-1.
 Prolate, of numbers 108, *see* Proof 370, ii. 533.
 Proportion: theory discussed Pythagoras 84-5, but 1 numerical and applicables mensurables only 153, def. of numerical proportions the 'perfect' proportions Euclid's universally accepted theory due to Eudoxus 216, 325-7.
 Proposition, geometrical divisions of, 370-1.
 Protagoras 202: on mathematics 23, 179.
 Prou, V. ii. 309.
 Psammites or Sand-reckoner medes 40, ii. 3, ii. 81-5.
 Psellus, Michael, 223-4 n., ii. 545-6.
 Pseudaria of Euclid 430-1.
 Pseudo-Boëtius 47.
 Pseudo-Eratosthenes: letter of publication of cube 244-5.
 Ptolemies: coins of, with astronomical numerals 34-5: Ptolemy of, 354.
 Ptolemy, Claudius, 181, ii. 216, ii. 218, ii. 273-4: decimal fractions 44-5: approximation to π 233: attempt parallel-postulate 358, *Syntaxis* ii. 273-86, commentaries and editions ii. 274-5, of, ii. 275-6, trigonometric tables 276-86, 290-1, Table of chords ii. 259, ii. 283-4, on oblique ecliptic ii. 107-8: *Geographia* ii. 286-92: *Planisphaerium* 3, *Optics* ii. 293-4, other works 293: *περὶ ὁρίων* ii. 295: *σάρασις* *ib.*
 Pyramids: origin of names measurements of, in Rhinoceros 126-8: pyramids of

- Gizeh, and Mēdūm 128: measurement of height by Thales 129-30; volume of pyramid 176, 180, 217, ii. 21, &c., volume of frustum ii. 334.
- Pythagoras 65-6, 121, 131, 133, 138: travels 4-5, story of bribed pupil 24-5: motto 25, 141: Heraclitus, Empedocles and Herodotus on, 65: Proclus on discoveries of, 84-5, 90, 119, 141, 154: made mathematics a part of liberal education 141, called geometry 'inquiry' 166, used definitions 166: arithmetic (theory of numbers) 66-80, figured numbers 76-9: gnomons 77, 79: 'friendly' numbers 75: formula for right-angled triangles in rational numbers 79-80: founded theory of proportion 84-5, introduced 'perfect' proportion 86: discovered dependence of musical intervals on numerical ratios 69, 75-6, 85, 165: astronomy 162-3, earth spherical *ib.*, independent movement of planets 67, 163: Theorem of Pythagoras 142, 144-9, how discovered? 147-9, general proof, how developed *ib.*, Pappus's extension ii. 369-71.
- Pythagoreans 2, 11, 220: *quadrivium* 11: a Pythagorean first taught for money 22: first to advance mathematics 66: 'all things are numbers' 67-9: 'number' of an object 69, 'number in the heaven' 68: figured numbers 69: definition of unit 69: 1 is odd-even 71: classification of numbers 72-4: 'friendly' numbers 75: 10 the 'perfect' number 75: oblong numbers 82-3, 108, 114: side- and diameter-numbers giving approximations to $\sqrt{2}$, 91-3: first cases of indeterminate analysis 80, 91, 96-7: sum of angles of triangle = $2R$, 135, 143: geometrical theorems attributed to 142, 54, 163, 164, 165, 166, 167, 216: construction of regular pentagon 160-2: astronomical system (non-geocentric) 163-5: definitions 166: on order of planets ii. 242.
- Gay en heru*, height (of pyramid) 127.
- Quadratic equation: solved by Pythagorean application of areas 150-2, 167, 394-6, 422-3: numerical solutions ii. 344, ii. 448, ii. 463-5.
- Quadratrix* 2, 23, 171, 182, 218, 219, 225-30, ii. 379-82.
- Quadrivium* of Pythagoreans 11.
- Quinary system of numerals 26.
- Quintilian ii. 207.
- Qustā b. Lūqā, translator of Euclid 362, ii. 453.
- Rangabé, A. R. 49-50.
- Ratdolt, Erhard, first edition of Euclid 364-5.
- Reductio ad absurdum* 372: already used by Pythagoreans 168.
- Reduction* (of a problem) 372.
- Reflection: equality of angles of incidence and reflection 442, ii. 294, ii. 353-4.
- Refraction 6-7, 444: first attempt at a law (Ptolemy) ii. 294.
- Regiomontanus 369, ii. 27, ii. 453-4.
- Regula Nicomachi* 111.
- Rhabdas, Nicolas, 40, ii. 324 *n.*, ii. 550-8.
- Rhind Papyrus: mensuration in, 122-8: algebra in, ii. 440-1.
- Right-angled triangle: inscribed by Thales in circle 131: theorem of Eucl. I. 47, attributed to Pythagoras 142, 144-5, supposed Indian origin of, 145-6.
- Right-angled triangles in rational numbers: Pythagoras's formula 80, Plato's 81, Euclid's 81-2, 405: triangle (3, 4, 5) known to Egyptians 122: Indian examples 146: Diophantus's problems on, ii. 527-14.

Rüstow, F. W. ii. 309.
Ruler-and-compasses restriction
175-6.

Sachs, Eva, 209 n.
Salaminian table 48, 50-1.
Salinon ii. 23, ii. 103.
Sampi ($\gamma = 900$) derived from
Ssade q.v.
Sar. (Babylonian for 60^2) 28, ii. 215.
Satapatha Brāhmaṇa, 146.
Savile, Sir H., on Euclid 360, 369.
Scalene: of triangles 142: of certain
solid numbers 107: of an odd
number (Plato) 292: of an oblique
cone ii. 134.
Schiaparelli, G. 317, 330, ii. 539.
Schmidt, W. ii. 308, 309, 310.
Schöne, H. ii. 308.
Schöne, R. ii. 308, 317.
Scholiast to *Charmides* 14, 53.
Schooten, F. van, 75 n., ii. 185.
Schulz, O. ii. 455.
Scopinas ii. 1.
Secondary numbers 72.
Sectio canonis 17, 215, 444.
Seelhoff, P. 75 n.
Seleucus ii. 3.
Semicircle: angle in, is right
(Thales) 131, 133-7.
Senkereh, Tables 28, 29.
Senti, base (of pyramid) 127.
Se-geť, 'that which makes the nature'
(of pyramid) = cotangent of angle
of slope 127-8, 130, 131.
Serenus ii. 519-26: *On section of
cylinder* ii. 519-22, *On section of
cone* ii. 522-6.
Sesostris (Ramses II) 121.
Sexagesimal system of numerals
and fractions 28-9: sexagesimal
fractions in Greek 44-5, 59, 61-3,
233, ii. 277-83.
Sextius 220.
Sicily 8.
'Side-' and 'diameter-numbers' 91-
3, 112, 153, 308, 380, ii. 536.
Simon, M. 200.
Simplicius: extract from Eudemus
on Hippocrates's quadrature of
lunes 171, 182-99: on Antiphan
221-2: on Eudoxus's theory of
concentric spheres 329: commen-
tary on Euclid 358, ii. 539-40: on

Simon, R., edition of
Elements 365, 369, and of
Data 421: on Euclid's
435-6: restoration of *Plato*
of Apollonius ii. 185, ii.
Simus of Posidonia 86.
Sines, Tables of, ii. 253, ii.
Sinus rectus, sinus versus 36.
Sluse, R. F. de, 96.
Smith, D. E. 49, 133 n.
'Solid' loci and problems
117-18: *Solid Loci* of *A*
438, ii. 118-19.
'Solid' numbers, classified
Solids, Five regular: discov-
tributed to Pythagoras c.
goreans 84, 141, 158-
alternatively (as regard-
dron and icosahedron) t.
tetus 162: all five inv-
by Theaetetus 159, 162, 2.
Plato on, 158-60: Eucl-
structions for, 415-19: 1.
constructions ii. 368-9: 1.
of, ii. 335, ii. 395-6.
Solon 4, 48.
Sophists: taught mathema-
Sosigenes 316, 329.
Soss = *sussu* = 60 (Babylonian)
ii. 215.
Speusippus 72, 73, 75, ii.
Pythagorean numbers
on the five regular solids
theorems ib.
Sphaeric 11-12: treatises o-
tolycus and Euclid 348-
1: earlier text-book pre-
in Autolycus 349-50: 2.
of Theodosius ii. 245, 246.
Menelaus ii. 252-3, 260,
Sphere-making 18: Archim-
ii. 17-18.
Spiric sections ii. 203-6.
Sporus 226: criticisms on
trix 229-30: *κρηία* 234: 1.
tion of cube 266-8.
Square root, extraction o-
ex. in sexagesimal
(Theon) 61-2, (scholias-
clid) 63: method of ap-
ting to surds ii. 51-2, i.
ii. 547-9, ii. 553-4.
Square numbers 69: form-
adding successive gnom-

- numbers) 77 : any square is sum of two triangular numbers 83-4 : 8 times a triangular number + 1 = square, 84, ii. 516.
- Ssade*, Phœnician sibilant (signs T A M P) became \aleph (900) 32.
- 'Stadium,' 1/60th of 30°, ii. 215.
- Stadium* of Zeno 276-7, 281-3.
- Star-pentagon, or *pentagram*, of Pythagoreans 161-2.
- Stereographic projection (Ptolemy) ii. 292.
- Stevin, S. ii. 455.
- 'Stigma,' name for numeral 5, originally Σ (digamma) 32.
- Strabo 121, ii. 107, ii. 220.
- Strato ii. 1.
- Subcontrary (= harmonic) mean, defined 85.
- Subtraction in Greek notation 52.
- Surds : Theodorus on, 22-3, 155-6, 203-9, 304 : Theaetetus's generalization 203-4, 205, 209, 304 : see also 'Approximations'.
- Surface-Loci* 219, ii. 380-5 : Euclid's 439-40, ii. 119, ii. 425-6.
- Sūrya-Siddhānta* ii. 253.
- Sussu* = *soos* (Babylonian for 60) 28, ii. 215.
- Synesiuss of Cyrene ii. 293.
- Synthesis 371-2 : defined by Pappus ii. 400.
- Syracuse 8.
- Table of Chords 45, ii. 259-60, ii. 283.
- Tāittirīya Samhitā* 146.
- Tannery, P. 15, 44, 87, 89, 119, 132, 180, 182, 184, 188, 196 n., 232, 279, 326, 440, ii. 51, ii. 105, ii. 204-5, ii. 215, ii. 218, ii. 253, ii. 317, ii. 453, ii. 483, ii. 519, ii. 538, ii. 545, 546, ii. 550, ii. 556, ii. 561.
- Teles on secondary education 21.
- Teos inscription 32, 34.
- Tetrads of Apollonius 40.
- Tetrahedron : construction 416, ii. 368 : volume of, ii. 335.
- Thales 2, 4, 67 : one of Seven Wise Men 128, 142 : introduced geometry into Greece 128 : geometrical theorems attributed to, 130-7 : measurement of height of pyramid 129-30, and of distance of ship from shore 131-3 : definition of number 69 : astronomy 137-9, ii. 244 : predicted solar eclipse 137-8.
- Theaetetus 2, 119, 170 : on surds 22-3, 155, 203-4, 205, 209, 304 : investigated regular solids 159, 162, 212, 217 : on irrationals 209-12, 216-17.
- Themistius 221, 223, 224.
- Theodorus of Cyrene : taught mathematics 22-3 : on surds 22-3, 155-6, 203-9, 304.
- Theodosius ii. 245-6 : *Sphaerica* 349-50, ii. 246-52 : other works ii. 246 : no trigonometry in, ii. 250.
- Theologumena arithmetica* 96, 97, 318.
- Theon of Alexandria : examples of multiplication and division 58, 59-60 : extraction of square root 61-3 : edition of Euclid's *Elements* 360-1, ii. 527-8 : of *Optics* 441, ii. 528 : *Catoptrica* *ib.* : commentary on *Syntaxis* 58, 60, ii. 274, ii. 526-7.
- Theon of Smyrna 12, 72, 73, 74, 75, 76, 79, 83, 87, ii. 515 : treatise of, ii. 238-44 : on 'side-' and 'diameter-numbers' 91-3, 112 : forms of numbers which cannot be squares 112-13.
- Theophrastus 158, 163 : on Plato's view of the earth 315.
- Theudius 320-1.
- Theuth, Egyptian god, reputed inventor of mathematics 121.
- Thévenot, M. ii. 308.
- Thrasyllus 97, 176, 177, ii. 241, ii. 243.
- Thucydides ii. 207.
- Thymaridas : definition of unit 69 : 'rectilinear' = prime numbers 72.

(Dionysodorus and Heron), ii. 218-19, ii. 334-5.

Torelli, J. ii. 27.

Transversal: Menelaus's theorem for spherical and plane triangles ii. 266-70: lemmas relating to quadrilateral and transversal (Pappus) ii. 419-20.

'Treasury of Analysis' 421, 422, 439, ii. 399-427.

Triangle: theorem about sum of angles Pythagorean 135, 143, Geminus and Aristotle on, 135-6.

Triangle, spherical: called *τρίπλευρον* (Menelaus) ii. 262: propositions analogous to Euclid's on plane triangles ii. 262-5: sum of angles greater than two right angles ii. 264.

Triangular numbers 15, 69: formation 76-7: 8 times triangular number + 1 = a square 84, ii. 516.

Trigonometry ii. 5, ii. 198, ii. 257-9, ii. 265-73, ii. 276-86, ii. 290-1.

Trisection of any angle: solutions 235-44: Pappus on, ii. 385-6.

Tschirnhausen, E. W. v., 200.

Tycho Brahe 317, ii. 2, ii. 196.

Tzifra (= 0) ii. 547.

Ukha-thebt (side of base in pyramid) 126, 127.

Unit: definitions (Pythagoreans, Euclid, Thymaridas, Chrysippus) 69.

Usener, H. 184, 188.

Valla, G.: translator of extracts from Euclid 365, and from Archimedes ii. 26.

Venatorius, Thomas Ge
princeps of Archimede

Venturi, G. ii. 308.

Vieta 200, 223, ii. 182, ii. 557.

Vigesimal system (of nu

Vincent, A. J. H. 50, 4

ii. 545, ii. 546.

Vitruvius 18, 147, 174,

ii. 245: Vitruvius a

ii. 302-3.

Viviani, V. ii. 261.

Vogt, H., 156 n., 203 n.

Wescher, C. ii. 309.

Wilamowitz - Moellendo

158 n., 245, ii. 128.

Xenocrates 24, 319:
Numbers 319: upheld
lines' 181.

Xenophon, on arithmeti
tion 19.

Xylander (W. Holzmann
ii. 545.

Yahyā b. Khālīd b. Barn

Zamberti, B., translator
365, 441.

Zeno of Elea 271-3: arg
motion 273-83.

Zeno of Sidon on Eucl. I
221-2.

Zenodorus ii. 207-13.

Zero in Babylonian no
O in Ptolemy 39, 45.

Zeuthen, H. G. 190, 206
398, 437, ii. 52, ii. 1

ii. 290-1, ii. 405, ii. 4

Zodiac circle: obliquity
by Oenopides 138, 174

